Math 217 (101): Practice Final Solutions

1. Since we're asked to compute the flux of $\nabla \times \mathbf{F}$, it is probably best to use Stokes' theorem. Let S be the hemisphere (upper hemisphere of the sphere of radius 2, centred at the origin). The boundary of S is a circle, C, of radius 2 in the xy-plane (with counter-clockwise orientation if viewed from above). By Stokes,

$$\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Now there are (at least) two ways to proceed. The first way is to compute the line integral directly: paramterize C by $x(t) = 2\cos(t)$, $y(t) = 2\sin(t)$, $z(t) = 0$, $0 \leq t \leq 2\pi$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 0, 2\cos(t), 2\sin(t)\rangle \cdot \langle -2\sin(t), 2\cos(t), 0\rangle dt = 4\int_0^{2\pi} \cos^2(t)dt = 4\pi.$$

The second way is observe that the disk, D, bounded by C, with (upward) unit normal $\mathbf{n} = \mathbf{k}$, also has boundary C. So By Stokes (again)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} dS.$$

The third component of $\nabla \times \mathbf{F}$ is 1, so this surface integral is $area(D) = \pi(2)^2 = 4\pi$.

2a. Using the two equations, we see that $y = x^2$, and $z = (2/3)xy = (2/3)x^3$. Thus C is parameterized by $\mathbf{r}(x) = \langle x, x^2, (2/3)x^3\rangle$.

2b. On this part of the curve, x ranges from $x = 0$ to $x = 3$. So using the formula for arc length,

$$L = \int_0^3 |\mathbf{r}'(x)|dx = \int_0^3 \sqrt{1 + (2x)^2 + (2x^2)^2}dx$$

$$= \int_0^3 \sqrt{(2x^2+1)^2}dx = \int_0^3 (2x^2+1)dx = (2x^3/3 + x)|_0^3 = 21.$$

3. The disk is a closed, bounded region, so by the Extreme Value Theorem, f attains its maximum and minimum on the disk. We first look for critical points in the disk. Compute

$$f_x = \frac{2 + y^2 - x^2 + 2xy}{(2 + x^2 + y^2)^2}$$

and

$$f_y = \frac{-2 + y^2 - x^2 - 2xy}{(2 + x^2 + y^2)^2}.$$

For both partials to be 0, we must have $2 + y^2 - x^2 + 2xy = 0$ and $-2 + y^2 - x^2 - 2xy = 0$. Adding these two equations gives $x^2 = y^2$, so $y = \pm x$. Substituting this back into one of the equations gives $2 \pm 2x^2 = 0$. Thus the only the minus sign works, and $x = \pm 1$. So the critical points are $(1, -1)$ and $(-1, 1)$ (which both lie in the disk). Note $f(1, -1) = 1/2$, $f(-1, 1) = -1/2$. Now we check the boundary of the disk: the circle $x^2 + y^2 = 4$. It is helpful to note that on the boundary, $f(x, y) = (x - y)/6$, so we just have to find max/min of this function. There are several ways to do this. One is to parameterize the boundary – say, $x = 2\cos(t)$, $y = 2\sin(t)$, $0 \le t < 2\pi$. Then we have to find max/min of $g(t) = f(2\cos(t), 2\sin(t)) = (\cos(t) - \sin(t))/3$. So we solve $0 = g'(t) = -(\sin(t) + \cos(t))/3$ to get $t = 3\pi/4$ or $t = 7\pi/4$. Now $g(3\pi/4) = -2(1/\sqrt{2})/3 = -\sqrt{2}/3$ and $g(7\pi/4) = \sqrt{2}/3$. Which is bigger, $\sqrt{2}/3$ or $1/2$? To see, square them: $(1/2)^2 = (1/4) = 9/36$ and $(\sqrt{2}/3)^2 = 2/9 = 8/36$. So $1/2$ is bigger. Thus the max/min values of f on the disk are $\pm 1/2$. A second way to proceed is Lagrange multipliers: to min/max $x - y$ subject to $x^2 + y^2 = 4$, solve $\nabla(x - y) = \lambda \nabla(x^2 + y^2)$ or $\langle 1, -1 \rangle = \lambda \langle 2x, 2y \rangle$. This gives $2x = (1/\lambda) = -2y$ so $x = -y$. Then by the constraint, $4 = x^2 + y^2 = 2x^2$ so $x = \pm\sqrt{2}$ and $f = (x - y)/6 = \pm 2\sqrt{2}/6 = \pm\sqrt{2}/3$.

4a. We can express the upper half of the region enclosed by S as

$$\{(x, y) \in D \mid 0 \le z \le 2\sqrt{1 - (x^2 + y^2)}\}$$

where D is the disk $x^2 + y^2 \le 1$ in the xy-plane. So the volume of the full region (both halves) is

$$V = 2 \int\int_D \int_0^{2\sqrt{1-(x^2+y^2)}} dz dA = 2 \int_0^{2\pi} \int_0^1 2\sqrt{1 - r^2} r dr d\theta$$

$$= 4\pi(-2/3)(1 - r^2)^{3/2}|_0^1 = 8\pi/3.$$

4b. Parameterize S by

$$\mathbf{r}(u, v) = \langle \sin(u)\cos(v), \sin(u)\sin(v), 2\cos(u) \rangle,$$

$0 \le u \le \pi$, $0 \le v \le 2\pi$. Then $r_u = \langle \cos(u)\cos(v), \cos(u)\sin(v), -2\sin(u) \rangle$ and $r_v = \langle -\sin(u)\sin(v), \sin(u)\cos(v), 0 \rangle$, so

$$r_u \times r_v = \langle 2\sin^2(u)\cos(v), 2\sin^2(u)\sin(v), \cos(u)\sin(u) \rangle.$$

Thus

$$|r_u \times r_v| = (4\sin^4(u) + \cos^2(u)\sin^2(u))^{1/2} = (4\sin^2(u) + \cos^2(u))^{1/2} \sin(u)$$

so

$$A = \int_0^{2\pi} \int_0^{\pi} (4\sin^2(u) + \cos^2(u)\sin^2(u))^{1/2} \sin(u) du dv$$

$$= 2\pi \int_0^{\pi} (4\sin^2(u) + \cos^2(u))^{1/2} \sin(u) du.$$

5. Taking the hint, let us add in the top of the cone, namely a horizontal disk, D, of radius H at $z = H$, with upward orientation. Let R be the closed surface $S \cup D$, and let E be solid region enclosed by R. By the Divergence theorem,

$$\int\int\int_E \nabla \cdot \mathbf{F} dV = \int\int_R \mathbf{F} \cdot d\mathbf{S} = \int\int_S \mathbf{F} \cdot d\mathbf{S} + \int\int_D \mathbf{F} \cdot d\mathbf{S}.$$

Now $\nabla \cdot \mathbf{F} = 1 + 2y + (1 + y) = 2 + 3y$. Using cylindrical coordinates (i.e., polar coordinates for x and y), we compute

$$\int\int\int_E (2 + 3y) dV = \int_0^{2\pi} \int_0^H \int_r^H (2 + 3r\sin(\theta)) r dz dr d\theta$$

$$= \int_0^{2\pi} \int_0^H (2 + 3r\sin(\theta))(H - r) r dr d\theta = \int_0^{2\pi} ((1/3)H^3 + (1/4)H^4 \sin(\theta)) d\theta$$

$$= ((1/3)H^3\theta - (1/4)H^4\cos(\theta))|_0^{2\pi} = 2\pi H^3/3$$

(note that we could have seen from the beginning that the triple integral of y would give 0, by symmetry). Now on D, $\mathbf{n} = \mathbf{k}$ so $\mathbf{n} \cdot \mathbf{F} = z + yz + x^3 = H + Hy + x^3$, and so (using polars)

$$\int\int_D \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^H (H + Hr\sin(\theta) + r^3\cos^3(\theta)) r dr d\theta$$

$$= \int_0^H (H\theta - Hr\cos(\theta) + r^3(\sin(\theta) - \sin^3(\theta)/3))|_0^{2\pi} r dr = \int_0^H 2\pi H r dr = \pi H^3$$

(again, it was obvious by symmetry that y and x^3 integrate to 0 over D, so we could have written down the answer – $H(\pi H^2) = \pi H^3$ – immediately). Finally,

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_E \nabla \cdot \mathbf{F} dV - \int\int_D \mathbf{F} \cdot d\mathbf{S} = (2/3 - 1)\pi H^3 = -\pi H^3/3.$$

6. S is the graph $z = f(x, y) = xy$ where (x, y) range over $D = \{(x, y) \mid x^2 + y^2 \leq 3\}$. Since $f_x = y$ and $f_y = x$, we have

$$I = \int\int_S (x^2 + y^2) dS = \int\int_D (x^2 + y^2)\sqrt{1 + x^2 + y^2} dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} r^2\sqrt{1 + r^2} r dr d\theta = 2\pi \int_0^{\sqrt{3}} r^3\sqrt{1 + r^2} dr$$

(using polar coordinates on D). Now let's integrate by parts with $u = r^2/2$, $dv = 2r\sqrt{1 + r^2}dr$ (so $du = rdr$ and $v = (2/3)(1 + r^2)^{3/2}$):

$$I = 2\pi[(1/3)r^2(1 + r^2)^{3/2}|_0^{\sqrt{3}} - (1/3)\int_0^{\sqrt{3}} 2r(1 + r^2)^{3/2} dr]$$

$$= 2\pi[8 - (1/3)(2/5)(1 + r^2)^{5/2}|_0^{\sqrt{3}}] = 2\pi[8 - (2/15)(32 - 1)]$$
$$= 4\pi[4 - 31/15] = 4\pi(60 - 31)/15 = (29/15)4\pi.$$

7. Differentiating $u = x^2 + xy - y^2$ with respect to u gives $1 = (2x+y)x_u + (x-2y)y_u$. At $(x, y) = (2, -1)$, this is $1 = 3x_u + 4y_u$. Differentiating $v = 2xy + y^2$ with respect to u gives $0 = 2yx_u + (2x+2y)y_u$. At $(x, y) = (2, -1)$, this is $0 = -2x_u + 2y_u$. So $y_u = x_u$. Plugging this into the first equation yields $1 = 7x_u$. Thus when $(x, y) = (2, -1)$, $\frac{\partial x}{\partial u} = 1/7$.

8a. **H** is conservative on $\mathbf{R}^3$ if and only if $\nabla \times \mathbf{H} = \mathbf{0}$. So compute

$$\nabla \times \mathbf{H} = \nabla\mathbf{F} + \lambda\nabla \times \mathbf{G} = \langle x, 0, -z\rangle + \lambda\langle x, 0, -z\rangle = (1 + \lambda)\langle x, 0, -z\rangle$$

and we see that **H** is conservative when $\lambda = -1$.

8b. Solve $f_x = 6x^2yz^2 - yz$ to get $f = 2x^3yz^2 - xyz + g(y, z)$. Next, solve $2x^3z^2 - xz + g_y = f_y = 2x^3z^2 + 2y - xz$ to get $g_y = 2y$, and so $g = y^2 + h(z)$. Finally, solve $4x^3yz - xy + h' = f_z = 4x^3yz - xy$ to get $h' = 0$, so $h = \mathit{constant}$ ($h = 0$ will do fine). So $f(x, y, z) = 2x^3yz^2 - xyz + y^2$ is a potential for **H** (when $\lambda = -1$).

8c. Let's be a bit clever here. Since **G** is simpler than **F**, we'll use the results of the previous two parts to get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\mathbf{F} - \mathbf{G} + \mathbf{G}) \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}.$$

Using the fundamental theorem for line integrals, the first line integral above is just $f(1, e, 1) - f(0, 1, 0) = 2e - e + e^2 - (1) = e^2 + e - 1$. To do the second line integral, we have to parameterize C. This is easily done using x as a parameter: $z(x) = x$, $y(x) = e^{xy(x)} = e^{x^2}$. So

$$\int_C \mathbf{G} \cdot d\mathbf{r} = \int_0^1 \langle xe^{x^2}, 0, xe^{x^2}\rangle \cdot \langle 1, 2xe^{x^2}, 1\rangle dx = \int_0^1 2xe^{x^2} dx = e^{x^2}|_0^1 = e - 1.$$

So $\int_C \mathbf{F} \cdot d\mathbf{r} = e^2 + e - 1 + e - 1 = e^2 + 2e - 2$.