Math 217: Vector Calculus (Ch. 17)

1 Ch. 17, Lecture 1 (Nov. 10)

17.1: Vector Fields

2D: A **vector field** on $D \subset \mathbb{R}^2$ is a function

$$\mathbf{F} : D \to \mathbb{R}^2 \ \ (= \text{ set of 2D vectors }).$$

Notation:

$$\begin{aligned}\mathbf{F}(x,y) &= \langle P(x,y), Q(x,y)\rangle = P(x,y)\hat{\mathbf{i}} + Q(x,y)\hat{\mathbf{j}} \\ &= P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}.\end{aligned}$$

Picture:

Physical examples: velocity of water on a lake surface, force field acting on a particle moving in the plane, etc.

3D: A **vector field** on $E \subset \mathbb{R}^3$ is a function

$$\mathbf{F} : E \to \mathbb{R}^3 \ \ (= \text{ set of 3D vectors }).$$

Notation:

$$\begin{aligned}\mathbf{F}(x,y,z) &= \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle \\ &= P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}Q\hat{\mathbf{k}}.\end{aligned}$$

Picture:

Physical examples: velocity of a fluid, force field acting on a particle moving in space, electric or magnetic field, etc.

Example: sketch $\mathbf{F}(x,y) = \langle x, y \rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

Example: sketch $\mathbf{F}(x,y,z) = \sin(z)\hat{\mathbf{k}}$.

One class of vector fields we are already familiar with is **gradient vector fields**.

If $f(x,y)$ is a function of 2 variables, then ∇f is a vector field:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = f_x\hat{\mathbf{i}} + f_y\hat{\mathbf{j}}.$$

The same holds for a function of 3 variables.

Definition: a vector field $\mathbf{F}$ is called **conservative** if it is a gradient; that is, if there is a (scalar) function f such that $\mathbf{F} = \nabla f$. In this case, f is called a **potential function** for $\mathbf{F}$.

Example: show $\mathbf{F}(x,y) = \langle 2xy, x^2 - 3y^2 \rangle$ is a conservative vector field.

17.2: Line Integrals

Goal: define $\int_C f ds$ and $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$, f is a (scalar) function, and $\mathbf{F}$ is a vector field.

Suppose $f(x, y)$ is a continuous function on $\mathbb{R}^2$, and C is a curve in $\mathbb{R}^2$ parameterized by a vector function

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \qquad a \leq t \leq b,$$

which is "smooth" (recall this means $\mathbf{r}'$ is continuous and $\mathbf{r}' \neq \mathbf{0}$).

We derive an expression for the line integral $\int_C f ds$ by constructing a Riemann sum:

Definition: The **line integral** of f over C is

$$\int_C f ds := \int_a^b f(x(t), y(t))[(x'(t))^2 + (y'(t))^2]^{1/2} dt \qquad \left(= \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)| dt\right).$$

Remark: This integral is independent of a choice of parameterization $\mathbf{r}(t)$ (which can be seen, for example, from the Riemann sum expression).

Remark: If $f \geq 0$, we can interpret the integral as the area of a "fence" of height $f(x, y)$ built over the curve C. If $f \equiv 1$, we recover the arc length of the curve.

Example: compute $\int_C x ds$ where C traverses the 1/4-circle $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$ once, counter-clockwise.

A physical interpretation: suppose C represents a wire with variable density $\rho(x, y)$. Then the mass of the wire is

$$m = \int_C \rho ds,$$

and the centre of mass is $(\bar{x}, \bar{y})$ with

$$\bar{x} = \frac{1}{m} \int_C x\rho ds, \qquad \bar{y} = \frac{1}{m} \int_C y\rho ds.$$

Two more kinds of line integral:

$$\int_C f dx := \int_a^b f(x(t), y(t)) x'(t) dt,$$

$$\int_C f dy := \int_a^b f(x(t), y(t)) y'(t) dt.$$

Example: compute $\int_C (y^2 dx + x dy)$ where (a) C is the straight line segment joining $(0, 0)$ and $(1, 1)$; and (b) C is the piece of the parabola $y = x^2$ joining the same two points.

Remark: the value of a line integral depends on the *path*, not just the endpoints (important exception coming soon!).

Remark: the direction in which the curve is traversed (called the "orientation") matters for line integrals with respect to x or y, but not for those "with respect to arc length", ds.

Line integrals of functions $f(x, y, z)$ of three variables over curves C in $\mathbb{R}^3$ are defined in the same way:

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Line integrals of vector fields:

Suppose a particle moves along a curve C in $\mathbb{R}^3$, and is subject to a force $\mathbf{F}(x, y, z)$ (a vector field). Let's compute the work done by the force on the particle:

Definition: Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C which is parameterized by $\mathbf{r}(t)$, $a \leq t \leq b$. The **line integral** of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} := \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \qquad (= \int_C \mathbf{F} \cdot \mathbf{T} ds).$$

Example: find the work done by the force $\mathbf{F}(x, y) = x\sin(y)\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ on a particle that moves along the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

Connection between line integrals of vector fields and scalar fields: let $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$. Then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t))\mathbf{r}'(t)dt = \int_a^b \langle P, Q, R\rangle \cdot \langle x', y', z'\rangle dt \\ &= \int_a^b (Px' + Qy' + Rz')dt = \int_C (Pdx + Qdy + Rdz).\end{aligned}$$

17.3: The Fundamental Theorem for Line Integrals

Recall, the FTC:

Theorem: Let C be a smooth curve described by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function (of 2 or 3 variables) for which ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Remark: So the value of a line integral of a conservative vector field ∇f is completely determined by the values of f at the endpoints. In particular, its value is independent of the path taken between the endpoints.

Proof:

Example: find the work done by $\mathbf{F}(x, y) = x^2y^3\hat{\mathbf{i}} + x^3y^2\hat{\mathbf{j}}$ moving an object from $(0, 0)$ to $(2, 1)$.

Terminology: a piecewise smooth curve will be called a **path**.

Independence of Path

Definition: Let $\mathbf{F}$ be a continuous vector field on a domain D. We say the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** in D if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D which share the same initial and terminal points.

Remark: So we know from the fundamental theorem that if $\mathbf{F}$ is conservative in D, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D.

Definition: We say a path C is **closed** if its initial and terminal points coincide.

Theorem: $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path in D.

Proof:

Remark: A physical consequence of this is that the work done by a conservative force around a closed path is zero.

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Last time, we saw that if $\mathbf{F}$ is a conservative vector field, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path (this follows from the Fundamental Theorem for line integrals). The converse is also true:

Theorem: Let D be an *open*, *connected* region, and let $\mathbf{F}$ be a continuous vector field on D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path on D, then $\mathbf{F}$ is conservative on D (i.e. there is f such that $\mathbf{F} = \nabla f$).

Proof:

Question: how do we check if a vector field $\mathbf{F}$ is conservative? We could (a) try to find f such that $\mathbf{F} = \nabla f$ (i.e. integrate), or (b) try to check if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path (forget it!). Is there an easier way? Yes!

Suppose $\mathbf{F} = \nabla f$ (in 2 dimensions for now), and compute:

So we have proved...

Theorem: if $\mathbf{F} = P(x,y)\hat{\mathbf{i}} + Q(x,y)\hat{\mathbf{j}}$ is conservative (and P, Q have continuous first partials), then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Now, the converse of this theorem is not always true. Consider,

Example: $\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ on $D = \mathbb{R}^2 \backslash \{(0,0)\}$.

Moral: we have to be careful about the domain. Precisely,

Theorem: Let D be an open, *simply-connected* region, and let $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$ where P, Q, have continuous first partials, and satisfy $\partial P/\partial y = \partial Q/\partial x$ in D. Then $\mathbf{F}$ is conservative.

Proof: later.
Example: is the vector field conservative? If so, find a potential function.

1. $\langle ye^x + \sin(y), e^x + x\cos(y) \rangle$
2. $(6x + 5y)\hat{\mathbf{i}} + (5x + 4y)\hat{\mathbf{j}}$.

Example: "conservation of energy".

17.4 Green's Theorem

Theorem: let C be a *positively oriented*, piecewise smooth, simple closed curve in the plane, and let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ have continuous partials on an open region containing D, then

$$\int_C (P dx + Q dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Remark: Green's theorem can be thought of as a version of the FTC for double integrals.

Proof of Green's theorem for "simple" regions D (those which are of Type I *and* Type II):

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Example: let C be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,2)$, with counter-clockwise orientation. Evaluate $\int_C (xydx + x^2y^3dy)$ directly, and using Green's theorem.

(Sketch of) proof of: $\partial Q/\partial x = \partial P/\partial y$ in D, simply connected, implies $\int_C \langle P, Q\rangle \cdot d\mathbf{r}$ independent of path in D (hence $\langle P, Q\rangle$ conservative in D).

Example: find the area inside the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Example: let $\mathbf{F} = \langle -y/(x^2+y^2), x/(x^2+y^2) \rangle$. Show $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for *any* simple closed path encircling the origin.

17.5: Curl and Divergence

Let $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ be a vector field on $\mathbb{R}^3$.

Definition: the **curl** of $\mathbf{F}$ is the vector field

$$curl\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}.$$

Definition: the **divergence** of $\mathbf{F}$ is the (scalar) function

$$div\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Remark on notation:

Example: find the divergence and curl of $\mathbf{F} = \langle xy, yz, xz \rangle$.

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Theorem: if $f(x, y, z)$ has continuous second partials, then

$$curl(\nabla f) = \mathbf{0}.$$

Proof:

Remark: this theorem says: $\mathbf{F}$ conservative $\implies curl\mathbf{F} = \mathbf{0}$.

Special case: $\mathbf{F} = \langle P(x, y), Q(x, y), 0\rangle$ (a $2D$ vector field "in disguise"):

Converse result:

Theorem: if $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ with continuous partials, and $curl\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative.

Proof: later.

Example: show $\mathbf{F} = 2xy\hat{\mathbf{i}} + (x^2 + 2yz)\hat{\mathbf{j}} + y^2\hat{\mathbf{k}}$ is conservative, and find a potential function for $\mathbf{F}$.

Theorem: suppose $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ has continuous second partials. Then

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Proof:

Example: show $\mathbf{F} = \langle xy^2, yz^2, x^2z \rangle$ is not a curl.

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17.6: Parametric Surfaces

Recall that we parameterize curves (one-dimensional objects) using vector functions $\mathbf{r}(t)$ of a single variable (parameter) t.

Similarly, we can parameterize surfaces (two-dimensional objects) using vector functions

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

of two variables (parameters), u and v.

Definition: the set

$$S = \{(x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3 \mid (u,v) \in D \subset \mathbb{R}^2\}$$

is called a **parametric surface**.

Example: parameterize the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Example: identify the surface parameterized by $\mathbf{r}(x,\theta) = \langle x, \cos(\theta), \sin(\theta)\rangle$, $0 \le \theta \le 2\pi$, $0 \le x \le 1$.

Example: parameterize a *surface of revolution*:

Tangent Planes

Differentiating $\mathbf{r}(u, v)$ with respect to u and v yields vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ which are tangent to the surface S:

So $\mathbf{r}_u \times \mathbf{r}_v$ is normal to the tangent plane.

Definition: we say a parametric surface S is **smooth** if $\mathbf{r}_u \times \mathbf{r}_v$ is never $\mathbf{0}$.

Example: find the tangent plane to the ellipsoid given by

$$\mathbf{r}(u,v) = \langle a\sin(u)\cos(v), b\sin(u)\sin(v), c\cos(u)\rangle,$$

at $(a,0,0)$.

Surface Area

Let's estimate the area of a parametric surface:

Definition: Let S be a smooth parametric surface parameterized by $\mathbf{r}(u, v)$, $(u, v) \in D$ (with S covered just once as (u, v) range over D). Then the **surface area** of S is

$$A(S) := \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA.$$

Example: find the surface area of the ellipsoid $x^2/a^2 + y^2/a^2 + z^2/c^2 = 1$.

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Special case (Section 16.6):

S = graph of a function $f(x, y)$:

$$S = \{(x, y, z) \mid (x, y) \in D, z = f(x, y)\}.$$

So

$$A(S) = \iint_D (1 + [f_x(x, y)]^2 + [f_y(x, y)]^2)^{1/2} dA$$

(and we recover the formula from Section 16.6).

17.7: Surface Integrals

Let $f(x, y, z)$ be a function of 3 variables, and let S be a surface. We can define the surface integral of f over S as follows:

So if S is parameterized by $\mathbf{r}(u, v)$, $(u, v) \in D$, we have:
Definition: The **surface integral** of f over S is

$$\iint_S f dS := \iint_D f(x(u, v), y(u, v), z(u, v))|\mathbf{r}_u \times \mathbf{r}_v| dA.$$

Example: if S is the graph $z = g(x, y)$, $(x, y) \in D$, then

$$\iint_S f dS = \iint_D f(x, y, g(x, y))\sqrt{1 + g_x^2 + g_y^2} dA.$$

Example: let $\rho(x, y, z)$ be the mass density of a thin sheet with shape S:

Example: find the centre of mass of the upper hemisphere of radius R (with constant density ρ_0).

Oriented surfaces

Definition: If it is possible to make a choice $\mathbf{n}(x, y, z)$ of unit normal vector at each point (x, y, z) on a surface S so that $\mathbf{n}$ varies continuously over S, then S is called an **oriented surface**, and the choice $\mathbf{n}(x, y, z)$ is called an **orientation** for S (there are 2 possible choices).

Example: (not all surfaces are orientable) Möbius strip.

Example: $S =$ graph of $f(x, y)$.

Example: sphere

Example: a parameterization gives an orientation

Convention: let S be a **closed** surface (that means S is the boundary of some solid region E). The **positive orientation** is the one for which the normal vectors point outward from E.

Surface Integrals of Vector Fields

Definition: let $\mathbf{F}$ be a continuous vector field, defined on an oriented surface S with unit normal $\mathbf{n}$. The **surface integral of F over** S (also called the **flux of F through** S) is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS.$$

A physical interpretation:

If S is a parametric surface, given by $\mathbf{r}(u, v)$, $(u, v) \in D$, then

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

is a unit normal, so

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[\mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v| dA = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA.$$

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Special case: S is the graph $z = f(x, y)$, $(x, y) \in D$.

Example: find the flux of $\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ across the piece of cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$ (oriented with upward pointing normal).

Physics: let $\mathbf{E}(x, y, z)$ be an electric field, S = a closed surface, and Q the total electric charge enclosed by S. Gauss Law:

$$Q = \epsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}.$$

Example: what is the total charge enclosed by a sphere of radius a centred at the origin, if the electric field is

$$\mathbf{E} = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \qquad (= \mathbf{r}/r^3\ )?$$

17.9: Stokes' Theorem

Stokes' theorem is a version of the fundamental theorem of calculus for surface integrals.

Theorem: Let S be a (piecewise smooth) oriented surface whose boundary is a simple, closed, (piecewise smooth) curve C, oriented "positively". Let $\mathbf{F}$ be a vector field whose components have continuous partials (on a domain containing S). Then

$$\iint_S curl\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Proof of Stokes' theorem in the special case when S is a graph:

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Example: find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle x^2 z, xy - 2xyz, y - xz \rangle = \nabla \times \langle xyz, xy, x^2 yz \rangle$, and S is as shown:

Example: Let C be the triangle with vertices $(1, 0, 0)$, $(0, 0, 1)$, and $(0, 1, 0)$ (visited in that order). Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (x + y^2)\hat{\mathbf{i}} + (y + z^2)\hat{\mathbf{j}} + (z + x^2)\hat{\mathbf{k}}$.

17.9: The Divergence Theorem

Theorem: (the “Divergence Theorem”, or “Gauss’s Theorem”) Let E be a solid region. Let S be the boundary surface of E, with positive (outward) orientation. Let $\mathbf{F}$ be a vector field (with continuous partials in a region containing E). Then

$$\iiint_E (div\mathbf{F})dV = \iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Proof of the divergence theorem for “simple” solid regions (time permitting):

...

Example: find the flux of $\mathbf{F} = \langle 3xy^2, xe^z, z^3 \rangle$ through the pictured surface:

Example: compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle z^2 x, y^3/3 + \tan z, x^2 z + y^2 \rangle$, and S is the unit upper-hemisphere (with "upward" pointing normal).

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Example: find the flux of $\mathbf{F} = x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ across the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane (with upward orientation).

Example: what is the flux of

$$\mathbf{F} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{3/2}}$$

across any closed surface?

Example: let $\mathbf{F} = \nabla \times \mathbf{G}$, where $\mathbf{G} = \langle x^2yz, yz^2, z^3e^{xy} \rangle$. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is as shown:

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Example: Let S be a closed surface bounding a solid region E. Let f be a function with continuous partials. Show

$$\iint_S (f\mathbf{n})dS = \iiint_E \nabla f dV$$

(note this is a *vector* equation).

Example: Let S be a surface whose boundary is a curve C, and let f and g be functions with continuous second partials. Show

$$\int_C f\nabla g \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}.$$