Math 217: Multiple Integrals (Ch. 16)

1 Ch. 16, Lecture 1 (Oct. 25)

16.1: Double Integrals over Rectangles

Recall the definition of the Riemann integral for a function of a single variable:

For a function of 2 variables, we define the **double integral** over a rectangle $R = [a, b] \times [c, d]$ analogously:

Definition:

$$\iint_R f(x, y)dA := \lim_{n,m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$$

(if this limit exists).

Theorem: if $f(x, y)$ is continuous on R, then this limit exists (and is independent of the choices of (x_{ij}^*, y_{ij}^*)).

Proof: omitted.

Definition: Let $f(x, y)$ be a continuous, non-negative function on a rectangle R. Then the **volume** under the graph of f is

$$V := \iint_R f(x, y)dA.$$

Remark on notation:

$$\iint_R f(x, y)dA = \iint_R f(x, y)dxdy = \iint_R f dA.$$

Example: approximate $\iint_{[0,1]\times[0,1]} \sqrt{1 - y^2} dA$ by a Riemann sum with $m = n = 2$, then find the exact value.

Midpoint rule for approximating double integrals:

$$\iint_R f(x, y)dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta x \Delta y$$

where $\bar{x}_i := (x_{i-1} + x_i)/2$, $\bar{y}_j := (y_{j-1} + y_j)/2$.
(This is what we just used!).

Average value

Recall the average value of a function in one-variable calculus:

Definition: The **average value** of a function $f(x, y)$ over the rectangle R is

$$f_{av} := \frac{1}{area(R)} \iint_R f(x, y) dA.$$

Example: the average value of $f(x, y) = \sqrt{1 - y^2}$ on $[0, 1]^2$ is:

Properties of double integrals:

- $\iint (f(x, y) + g(x, y)) dA = \iint f(x, y) dA + \iint g(x, y) dA$
- $\iint (cf(x, y)) dA = c \iint f(x, y) dA$
- $f(x, y) \geq g(x, y) \quad \Longrightarrow \quad \iint f(x, y) dA \geq \iint g(x, y) dA.$

The fact that these statements hold for integrals follows from the fact that they hold for the approximating Riemann sums. We do not give a proof, though.

16.2: Iterated Integrals

Question: how do we actually *compute* $\iint_R f dA$? (*Not* by the definition!)

Answer: by performing an iterated integral:

Theorem: (Fubini's Theorem). Suppose $f(x, y)$ is continuous on $R := [a, b] \times [c, d]$. Then

$$\iint_R f(x,y)dxdy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy = \int_a^b \left[\int_c^d f(x,y)dy\right] dx.$$

Example: Compute $\iint_{[0,1]^2} \sqrt{1-y^2}dA$.

Example: Sometimes one order is much easier than the other: compute $\iint_{[1,2]\times[0,\pi]} y\sin(xy)dxdy$.

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A special case of a function of two variables, is a function where the variables "separate": $f(x, y) = g(x)h(y)$. In this case, we have, for $R = [a, b] \times [c, d]$,

$$\iint_R g(x)h(y)dxdy = \int_a^b \left[\int_c^d g(x)h(y)dy\right] dx = \int_a^b g(x) \left[\int_c^d h(y)dy\right] dx = \left[\int_a^b g(x)dx\right] \left[\int_c^d h(y)dy\right].$$

Example: find the average value of x^2y over $[-1, 1] \times [0, 5]$.

16.3: Double Integrals over General Regions

Question: how do we even *define* (let alone compute) $\iint_D f dA$ when D is not a rectangle?
Answer: one way is to enclose D in a rectangle R. Set

$$\tilde{f}(x, y) := \begin{cases} f(x, y) & (x, y) \in D \\ 0 & (x, y) \in R \backslash D \end{cases}$$

and *define*

$$\iint_D f(x, y)dA := \iint_R \tilde{f}(x, y)dA.$$

Remark: Note that $\tilde{f}$ is typically discontinuous (at the boundary of D), and so we cannot be sure the integral makes sense. Don't worry, it will make sense if D is "nice enough", as it will be below. In any case, we will certainly *not* use this definition to *compute* integrals.

Now we identify two kinds of regions we will be able to integrate over:

Type I:

Type II:

If $f(x, y)$ is continuous on a region

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

of Type I, then

$$\iint_D f(x, y) dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx.$$

Similarly, if $f(x, y)$ is continuous on a region

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

of Type II, then

$$\iint_D f(x, y) dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right] dy.$$

Example: Find $\iint_D (y^2 - x) dA$ where D is the region bounded by the parabolas $x = y^2$ and $x = 3 - 2y^2$.

Example: find the volume bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Example: Compute $I = \int_0^1 \int_{3y}^3 e^{x^2} dxdy$.

Some basic properties of double integrals:

- for functions f and g,

$$\iint_D [f+g]dA = \iint_D f dA + \iint_D g dA$$

- for a function f and a constant c

$$\iint_D cf dA = c \iint_D f dA$$

- if $f(x,y) \geq g(x,y)$ for all $(x,y) \in D$, then

$$\iint_D f dA \geq \iint_D g dA$$

- if $D = D_1 \cup D_2$, and D_1 and D_2 don't overlap (except possibly at their boundaries), then

$$\iint_D f dA = \iint_{D_1} f dA + \iint_{D_2} f dA$$

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$$\iint_D 1 dA = A(D) \;\; (:= area(D))$$

- if $m \leq f(x,y) \leq M$ for all $(x,y) \in D$, then

$$mA(D) \leq \iint_D f dA \leq MA(D).$$

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16.4: Double Integrals in Polar Coordinates

Recall: polar coordinates, and why they are useful:

How to compute a double integral using polar coordinates:

So if $f(x, y)$ is continuous on the "polar rectangle" R given by $a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, we have

$$\iint_R f(x, y)dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))r dr d\theta.$$

Key point: $dA = dxdy$ is replaced by $rdrd\theta$.

More generally, if $f(x, y)$ is continuous on the polar region

$$D := \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\},$$

then

$$\iint_D f(x, y) dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta)) r dr d\theta.$$

Example: find the volume inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

Example: find the area of one "loop" of $r = \cos(3\theta)$.

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16.5: Applications of Double Integrals

Density, Moments, and Centre of Mass

Suppose a "lamina" (infinitesimally thin plate) occupies a region D in the xy-plane, and has density (mass per unit area) $\rho(x, y)$ at $(x, y) \in D$. Then its **total mass** is

$$m = \iint_D \rho(x, y) dA.$$

Remark: We could also consider, for example $\rho =$ "charge density", so the integral yields the total electric charge. Or $\rho =$ probability density..., etc.

We also define the **moment about the x-axis**

$$M_x := \iint_D y\rho(x, y) dA$$

and the **moment about the y-axis**

$$M_y := \iint_D x\rho(x, y) dA.$$

Then the **centre of mass** of the lamina is the point $(\bar{x}, \bar{y})$ with

$$\bar{x} = \frac{1}{m} M_y; \qquad \bar{y} = \frac{1}{m} M_x.$$

Remark:

- the centre of mass is the place to put your finger to balance the lamina horizontally
- when $\rho(x, y) \equiv 1$, we call the mass the "area", and the centre of mass the "centroid".

Example: find the centre of mass of the 1/4-disk $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$, if the density is proportional to

1. the distance from the x-axis
2. the square of the distance from the origin.

Remark: for more applications of double integrals (to moments of inertia, and probability, for example) see your text

16.6: Surface Area: we will skip this for now, and return to it when we hit Section 17.6.

16.7: Triple Integrals

Let $f(x, y, z)$ be a continuous function defined on a box $B = [a, b] \times [c, d] \times [r, s]$. We again define $\iiint f(x, y, z)dV$ as a limit of Riemann sums:

Theorem: (Fubini's Theorem). Suppose $f(x, y, z)$ is continuous on $B := [a, b] \times [c, d] \times [e, f]$. Then

$$\iiint_V f(x, y, z)dV = \int_a^b \left[\int_c^d \left[\int_r^s f(x, y, z)dz \right] dy \right] dx = \int_a^b \left[\int_r^s \left[\int_c^d f(x, y, z)dy \right] dz \right] dx = \cdots etc.$$

Triple integration over more general regions, E:

Type I:

Type II:

Type III:

Example: compute $\iiint_E z dV$, where E is bounded by $x = 0$, $y = 0$, $z = 0$, $y + z = 1$, and $x + z = 1$.

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Some applications of triple integrals.

Suppose a solid occupies the region $E \subset \mathbb{R}^3$ and has density $\rho(x, y, z)$. Then its mass is

$$m := \iiint_E \rho dV,$$

its moments are

$$\begin{aligned} M_{yz} &:= \iiint_E x\rho dV \\ M_{xz} &:= \iiint_E y\rho dV \\ M_{xy} &:= \iiint_E z\rho dV, \end{aligned}$$

and its centre of mass ("centroid" if $\rho \equiv 1$) is $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{xz}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}.$$

Example: find the volume and centroid of the solid bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, $x = 1$.

16.8: Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical coordinates (see 13.7):

Spherical coordinates (see 13.7):

Integration in cylindrical coordinates (it's just like polar!).

Suppose E is a "Type I" region:

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

and D can be nicely described in polar coordinates:

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}.$$

Then

$$\begin{aligned}\iiint f dV &= \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz\right] dA \\ &= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta), r\sin(\theta))}^{u_2(r\cos(\theta), r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z) r dz dr d\theta.\end{aligned}$$

Example: Compute $\iiint_E (x^3 + xy^2) dV$ where E is the region in the first octant beneath the paraboloid $z = 1 - x^2 - y^2$.

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Integration in spherical coordinates

Main point:

$$dV = dxdydz \to \rho^2 \sin(\phi) d\rho d\theta d\phi.$$

Why?:

Conclusion: Suppose, for example, that

$$E = \{(\rho, \theta, \phi) \mid \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\}.$$

Then

$$\iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_c^d \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho \sin(\phi)\cos(\theta), \rho \sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2 \sin(\phi) d\rho d\phi d\theta.$$

Example: compute $\iiint_H (x^2 + y^2) dV$ where H is the upper hemisphere $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$.

Example: find the volume of the solid solid lying within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \alpha\sqrt{x^2 + y^2}$ (α a positive constant).

16.9: Change of Variables

Recall the one-variable situation:

For functions of two variables, the setting is as follows:

Definition: the **Jacobian** of the transformation

$$T : (u, v) \mapsto (x(u, v), y(u, v))$$

is the function

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Change of variables formula:

Suppose T is a $1-1$ transformation with continuous partials and non-zero Jacobian which maps $S \subset uv-$plane onto $R \subset xy0$plane, and suppose f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv.$$

Example: Polar coordinates:

Example: evauluate $\iint_R e^{(x+y)/(x-y)} dA$ where R is bounded by $x = 0$, $y = 0$, $y + 1 = x$, $y + 2 = x$.

Idea behind the Jacobian factor: