**Math 217: Partial Derivatives (Ch. 15)**

# 1 Ch. 15, Lecture 1 (Sept. 24)

### 15.1: Functions of Several Variables

Functions of 2 variables

$$\begin{aligned} f :& D \subset \mathbb{R}^2 \to \mathbb{R} \\ &(x, y) \mapsto f(x, y) \end{aligned}$$

$D$ is the **domain** of $f$
The **range** of $f$ is

$$Ran(f) := \{f(x, y) \mid (x, y) \in D\} \subset \mathbb{R}$$

*Example*:

$$f(x, y) = \frac{1}{\sqrt{1 - x^2 - y^2/4}}$$

a) What is the domain of f?

b) What is the range of $f$?

*Definition*: The **graph** of a function $f$ defined on domain $D$ is the set

$$\{(x, y, f(x, y)) \mid (x, y) \in D\} \subset \mathbb{R}^3.$$

*Example*: Sketch the graph of the function $f$ above

*Example*: What is the graph of the function

$$f(x, y) = ax + by + d$$

("linear function" – here $a$, $b$, $d$ are given numbers)?

<u>Level Curves</u>

*Definition*: the **level curves** of a function $f$ of 2 variables are the curves

$$f(x, y) = k \qquad (k \text{ a constant })$$

in the $xy$-plane.

*Example*:

1) plane: $f(x, y) = ax + by + d$

2) $f(x, y) = x^2/4 + y^2$

## 2 Ch. 15, Lecture 2 (Sept. 27)

Functions of 3 (or more) variables

$$\begin{aligned} f : & D \subset \mathbb{R}^3 \to \mathbb{R} \\ & (x,y,z) \mapsto f(x,y,z) \end{aligned}$$

To draw the graph $w = f(x,y,z)$, we need 4 dimensions!

One thing we can do: sketch the **level surfaces** given by the equations

$$f(x,y,z) = k \qquad (k \text{ a constant. })$$

*Example*: What are the level surfaces of $f(x,y,z) = 3x^2 + 4y^2 + 5z^2$?

**15.2: Limits and Continuity**

*Example*: Consider the two functions

$$f(x,y) = \frac{x^4 - y^4}{x^2 + y^2}, \qquad g(x,y) = \frac{xy}{x^2 + y^2}$$

which are defined for all $(x,y) \in \mathbb{R}^2$ except the origin $(0,0)$. How do they behave as $(x,y) \to (0,0)$?

*Definition*: Suppose $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, and $(a,b) \in D$. We say that **the limit of** $f(x,y)$ **as** $(x,y)$ **approaches** $(a,b)$ **is** $L$, and write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

if for any number $\varepsilon > 0$, there is a number $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

Vector notation: write $\mathbf{x} = \langle x, y \rangle$, $\mathbf{x}_0 := \langle a, b \rangle$. Then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L$$

means that for any $\varepsilon > 0$, there is $\delta > 0$, such that $|f(\mathbf{x}) - L| < \varepsilon$ whenever $0 < |\mathbf{x} - \mathbf{x}_0| < \delta$.

*Example*: Find the limit, or show that it does not exist:

1) $\lim_{(x,y)\to(0,0)} \frac{xy+1}{x^2+y^2+1}$

2) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

3) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

<u>Continuity</u>

*Definition*: Suppose $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, and suppose $(a,b) \in D$. $f$ is **continuous at (a,b)** if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

$f$ is **continuous on D** if $f$ is continuous at each point in $D$.

*Example*:

1. polynomials are continuous:

2. rational functions are continuous on their domains:

3.

$$f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

4.

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

*Remark*: the notions of limit and continuity carry over to functions of $\geq 3$ variables in the obvious way.

# 3 Ch. 15, Lecture 3 (Sept. 29)

### 15.3: Partial Derivatives

Idea: a partial derivative of a function of several variables is obtained by treating all but one variable as a constant and differentiating in the usual way with respect to the remaining variable.

*Definition*: Suppose $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, and $(x, y) \in D$. The **partial derivative of $f$ with respect to $x$ at $(x, y)$** is

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

(if this limit exists). Similarly, The **partial derivative of $f$ with respect to $y$ at $(x, y)$** is

$$\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

(if this limit exists).

*Example*: Find $\partial/\partial x$ and $\partial/\partial y$ of $\frac{\sin(x^2+y^2)}{x^2}$.

Geometric picture:

*Example*: Find $\partial z/\partial x$ if $x^2 + y^2 + z^2 = 1$.

*Remark*: we can define and compute partial derivatives of functions of $\geq 3$ variables in the same way. We can also take higher partial derivatives:

*Example*: find the second partials of $f(x, y) = \cos(xy) - y^3$.

Note: $f_{xy} = f_{yx}$. This is not an accident!

*Theorem*: [equality of mixed partials] Suppose $f$ is defined on a some disk, $D \subset \mathbb{R}^2$, and suppose $(a,b) \in D$. If $f_{xy}$ and $f_{yx}$ both exist and are continuous on $D$, then $f_{xy}(a,b) = f_{yx}(a,b)$.

*Proof:* see appendix of text.

*Remark*: by this theorem, we also have, for example, $f_{xyy} = f_{yxy} = f_{yyx}$ (as long as these partials are continuous).

*Example*: $f(x,y) = \sin(x^2y)$:

**15.4 Tangent Planes and Linear Approximation**

Idea: close to a point on it, the graph of a "nice" function looks like a plane (just like in 1-dim. calculus the graph of a differentiable function, viewed up close, looks like a line – the tangent line!).

What is an equation for the tangent plane to the surface $z = f(x,y)$ at $(x_0, y_0, z_0)$?

*Definition*: If $f(x, y)$ has continuous partial derivatives, we call the plane given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

the **tangent plane to the surface** $z = f(x, y)$ **at** $(x_0, y_0, z_0)$.

*Example*: ellipsoid $x^2 + y^2/4 + z^2/9 = 1$.

# 4 Ch. 15, Lecture 4 (Oct. 1)

Linear Approximation

Recall that the tangent plane to the graph $z = f(x, y)$ at the point $(a, b, f(a, b))$ is given by the equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

This is the plane which best approximates the surface $z = f(x, y)$ at $(a, b, f(a, b))$. Equivalently, this is the linear function which best approximates the function $f$ near $(a, b)$.

*Definition*:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of $f$ at $(a, b)$, and the approximation $f(x, y) \approx L(x, y)$ is called the **linear approximation** or **tangent-plane approximation** to $f$ at $(a, b)$.

*Remark*: Compare this with the single variable case, where we have the "tangent line approximation":

$$f(x) \approx f(a) + f'(a)(x - a)$$

for $x$ near $a$.

*Example*:

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

So we see that the tangent plane may not be a "good" approximation to the graph of a function near a point.

Recall again the one-variable situation: $f(x)$ is "differentiable" at $x = a$ means

$$\frac{f(a + h) - [f(a) + f'(a)h]}{h} \to 0 \quad \text{as} \quad h \to 0.$$

In other words, the tangent line approximation $f(a + h) \approx f(a) + f'(a)h$ is a "good" one.

In the several variable case, we use the requirement that the tangent plane approximation be "good" as the *definition* of differentiability:

*Definition*: We say $f$ is **differentiable** at $(a, b)$ if

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f(a + \Delta x, b + \Delta y) - [f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y]}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0.$$

*Remark*: This definition is stated slightly differently in the text. But it is equivalent.

*Example*: So in the previous example, the function is *not* differentiable at $(0, 0)$ (even though the partials exist there).

The definition of differentiability is complicated to check for specific examples. Here is an easy criterion:

*Theorem*: if $f_x(x, y)$ and $f_y(x, y)$ exist for $(x, y)$ near $(a, b)$, and are continuous at $(a, b)$, then $f$ is differentiable at $(a, b)$.

*Proof*: Next time!

*Example*: show that for our example above, the partials are *not* continuous at $(0, 0)$.

*Example*: show that $f(x,y) = \sqrt{20 - x^2 - 7y^2}$ is differentiable at $(2,1)$. Find the linearization at that point, and use it to approximate $\sqrt{20 - (1.95)^2 - 7(1.05)^2}$.

We can re-phrase linear (tangent-plane) approximation in the language of "differentials": let $z = f(x,y)$, and let $\Delta z$ be the amount $z$ changes when we change $x$ by $\Delta x$ and $y$ by $\Delta y$. The linear approximation can be written

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x,y)\\ &\approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y - f(x,y)\\ &= f_x(x,y)\Delta x + f_y(x,y)\Delta y.\end{aligned}$$

Now thinking $(\Delta x, \Delta y) \to (0,0)$, and changing notation $\Delta x \mapsto dx$, $\Delta y \mapsto dy$, $\Delta z \mapsto dz$, we write the linear approximation as

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

This is just notation!

*Remark*: Of course, partial differentiation, linear approximation, and the notion of differentiability, all extend to function of $\geq 3$ variables in the obvious way.

For example,

$$f(x + \Delta x, y + \Delta y, z + \Delta z) \approx f_x(x,y,z)\Delta x + f_y(x,y,z)\Delta y + f_z(x,y,z)\Delta z.$$

# 5 Ch.15, Lecture 5 (Oct. 4)

### 15.5: Chain Rule

Some versions of the chain rule we already know:

1) real-valued functions of a single variable: if $g(t)$ is differentiable at $t = t_0$, and $f(x)$ is differentiable at $x = g(t_0)$, then $f(g(t))$ is differentiable at $t = t_0$ with

$$\frac{d}{dt} f(g(t))|_{t=t_0} = f'(g(t_0))g'(t_0)$$

2) vector-valued functions of 1 variable: if $s(t)$ (scalar function) is differentiable at $t = t_0$, and $\mathbf{r}(s)$ (vector function) is differentiable at $s = s(t)$, then $\mathbf{r}(s(t))$ is differentiable at $t = t_0$ with

$$\frac{d}{dt}\mathbf{r}(s(t))|_{t=t_0} = \mathbf{r}'(s(t_0))s'(t_0).$$

For functions of 2 variables, we have:

*Chain Rule (version 1)*: suppose $x(t)$ and $y(t)$ are differentiable at $t = t_0$, and suppose $f(x, y)$ is differentiable at $(x, y) = (x(t_0), y(t_0))$. Then the composite function $f(x(t), y(t))$ (a function of the single variable $t$) is differentiable at $t = t_0$, with

$$\frac{d}{dt} f(x(t), y(t))|_{t=t_0} = f_x(x(t_0), y(t_0))x'(t_0) + f_y(x(t_0), y(t_0))y'(t_0)$$
$$\left( = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right).$$

*Proof*:

*Example*: An insect is moving in the plane. Its position at time $t$ is given by $x(t) = \cos(t)$, $y(t) = 4\sin(t)$ (an ellipse!). The temperature at a point $(x, y)$ in the plane is $T(x, y) = e^{xy}$. What is the rate of change of temperature felt by the insect?

*Chain Rule (version 2)*: suppose $x(s,t)$ and $y(s,t)$ are differentiable functions of $s$, $t$, and suppose $f(x,y)$ is a differentiable function of $x$ and $y$. Then the composite function $f(x(s,t),y(s,t))$ is a differentiable function of $s$ and $t$, with

$$\frac{\partial}{\partial t}f(x(s,t),y(s,t)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$
$$\frac{\partial}{\partial s}f(x(s,t),y(s,t)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Diagram for keeping track of chain rule:

*Example*: Let $x = st$, $y = s^2t$, $z = st^2$, $f(x,y,z) = \tan(xy) + z^2$. Find $\partial f/\partial t$.

## 6 Ch.15, Lecture 6 (Oct. 8)

Implicit Differentiation

*Example*: If $x^4 + y^4 = 1$, find $dy/dx$.

Revisit this example: write the equation as $F(x, y) = 0$, where $F(x, y) := x^4 + y^4 - 1$. We *assume* that this equation defines $y = y(x)$ implicitly. Now use the chain rule:

(Justification of our "assumption": "Implicit Function Theorem":

if $F(x_0, y_0) = 0$, $F_x(x, y)$ and $F_y(x, y)$ are continuous near $(x_0, y_0)$, and $F_y(x_0, y_0) \neq 0$, then for $x$ near $x_0$, there is a differentiable function $y = y(x)$ with $y(x) = y_0$, and $F(x, y(x)) = 0$.)

*Example*: If $e^x + e^y + e^z = 1$, find $\partial z/\partial x$.

Revisit: $F(x, y, z) := e^x + e^y + e^z - 1 = 0$.

**15.6: Directional Derivatives and the Gradient**

*Definition*: Let $f(x, y)$ be a function of 2 variables, and let $\mathbf{u} =< a, b >$ be a unit vector. The **directional derivative** of $f$ at $(x_0, y_0)$ in the direction $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h},$$

if this limit exists.

*Remark*: $D_{\mathbf{u}} f(x, y)$ is the rate of change of $f$ in the direction $\mathbf{u}$. Note that $D_{\hat{\mathbf{i}}} f = \partial f/\partial x$ and $D_{\hat{\mathbf{j}}} f = \partial f/\partial y$.

*Definition*: The **gradient** of $f(x, y)$ is the vector-function (of 2 variables)

$$\nabla f(x, y) =< f_x(x, y), f_y(x, y) >= \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}.$$

*Remark*: the gradient is just a convenient way of recording the partial derivatives.

*Theorem*: if $f(x, y)$ is differentiable, then $f$ has a directional derivative in the direction $\mathbf{u}$ for any unit vector $\mathbf{u}$, and

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

*Proof:*

*Example*: find the rate of change of $f(x, y) = xy$ in the direction $\mathbf{u} = (\cos(\theta), \sin(\theta))$ for any $0 \leq \theta < 2\pi$. In what direction is $f$ changing the fastest at $(1, 1)$?

*Remark*: the gradient and directional derivatives carry over to functions of $\geq 3$ variables in the obvious way.

# 7 Ch.15, Lecture 7 (Oct. 13)

*Question*: in what direction is a function $f$ increasing/decreasing most rapidly?

*Answer*:

*Theorem*: suppose $f$ is differentiable. Then $D_{\mathbf{u}}$ is maximized when $\mathbf{u}$ points in the direction of $\nabla f$.

Re-phrase: $\nabla f$ points in the direction in which $f$ is most rapidly increasing.

*Example*: find the maximum rate of change of $f(x, y, z) = \frac{x}{y} + \frac{y}{z}$ at $(4, 2, 1)$, and the direction in which it is attained.

Tangent Planes to Level Surfaces

Let $F(x, y, z)$ be a function of 3 variables. Let $S$ be the level surface given by $F(x, y, z) = k$ (a two-dimensional surface in $\mathbb{R}^3$).

Let

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

describe a path on $S$, with $\mathbf{r}(0) = \langle x(0), y(0), z(0) \rangle = \langle x_0, y_0, z_0 \rangle$. Differentiate $F(x(t), y(t), z(t)) = 0$ with respect to $t$ at $t = 0$:

$$\begin{aligned} 0 &= \frac{d}{dt} F(x(t), y(t), z(t))|_{t=0} \\ &= F_x(x_0, y_0, z_0)x'(0) + F_y(x_0, y_0, z_0)y'(0) + F_z(x_0, y_0, z_0)z'(0) \\ &= \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(0) \end{aligned}$$

(here we used the chain rule). Recall that the vector $\mathbf{r}'(0)$ is tangent to $S$.

Conclusion: the gradient $\nabla F(x_0, y_0, z_0)$ is normal to the tangent plane to $S$ at $(x_0, y_0, z_0)$. In other words, the equation of the tangent plane to $S$ at $(x_0, y_0, z_0)$ is

$$\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$

Also, the **normal line** to $S$ at $(x_0, y_0, z_0)$ (the line through $(x_0, y_0, z_0)$ in the direction perpendicular to the tangent plane to $S$) is given by

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

*Example*: Let $S$ be the graph $z = f(x, y)$. Reproduce our familiar equation for the tangent plane.

*Example*: Find the tangent plane and normal line at a point on the sphere $x^2 + y^2 + z^2 = r^2$.

The situation in 2 dimensions is similar: $\nabla F(x, y)$ is perpendicular to the level curve $f(x, y) = k$.

Summarize properties of the gradient:

- $\nabla f$ points in the direction of fastest increase of $f$
- $\nabla f$ is perpendicular to the level curves/surfaces of $f$.

# 8 Ch.15, Lecture 8 (Oct. 15)

### 15.7: Maximum and Minimum Values

Sample problem to keep in mind: find the dimensions to minimize the cost of an aquarium (rectangular cube, open top) with a slate base and glass sides, if the cost of slate is 5 times the cost of glass.

Recall: max/min value situation in one dimension:

*Definition*: Let $f : D \subset \mathbb{R}^2 \to R$.

1. $f$ has a **local maximum** at $(a, b)$ if $f(x, y) \leq f(a, b)$ for all $(x, y)$ in some (possibly small) disk centred at $(a, b)$.
2. $f$ has a **local minimum** at $(a, b)$ if $f(x, y) \geq f(a, b)$ for all $(x, y)$ in some (possibly small) disk centred at $(a, b)$.
3. $f$ has an **absolute maximum** (resp. **absolute minimum**) at $(a, b)$ if $f(x, y) \leq f(a, b)$ (resp. $f(x, y) \geq f(a, b)$) for *all* $(x, y)$ in $D$.

*Theorem*: Suppose $f$ has a local maximum or minimum at $(a, b)$, and suppose $\nabla f(a, b)$ exists. Then $\nabla f(a, b) = \mathbf{0}$.
*Proof*:

*Remark*: $\nabla f(a, b) = \mathbf{0}$ means the tangent plane is horizontal.

*Definition*: We say $(a, b)$ is a **critical point** of $f$ if $\nabla f(a, b) = \mathbf{0}$, or if one of the partials fails to exist at $(a, b)$.

So we see that if $(a, b)$ is a local max./min. of $f$, then $(a, b)$ must be a critical point. The reverse implication is false in general.

*Example*: $f(x, y) = x^2 + 3y^2$

*Example*: $f(x, y) = xy$

*Theorem*: ("second derivatives test"). Suppose $\nabla f(a,b) =< 0,0 >$ (so $(a,b)$ is a critical point of $f$), and suppose that the second partials of $f$ are continuous on some disk centred at $(a,b)$. Set

$$D := f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

Then

- if $D > 0$ and $f_{xx}(a,b) > 0$, then $f$ has a local minimum at $(a,b)$
- if $D > 0$ and $f_{xx}(a,b) < 0$, then $f$ has a local maximum at $(a,b)$
- if $D < 0$, $f$ has neither a local max. nor a local min. at $(a,b)$.

*Remark*:

1. if $D < 0$, $(a,b)$ is called a **saddle point** of $f$
2. if $D = 0$, the test gives no information
3. note that

$$D = \det\begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{pmatrix}.$$

*Example*: Find and classify the critical points of $f(x,y) = y^2 - x^2$.

# 9 Ch.15, Lecture 9 (Oct. 20)

*Example*: Find and classify the critical points of $f(x, y) = x^2 + y^2 + x^2y + 4$.

*Example*: Find the dimensions of an aquarium (rectangular cube, open top) of given volume which minimizes the cost of the slate (for the base) and the glass (for the sides), supposing the slate costs 5 times as much as the glass.

Absolute maximum and minimum values

*Definition*:

- a set $D$ is called **bounded** if it can be contained in a disk (2D) or ball (3D)
- a set $D$ is called **closed** if it contains all of its boundary points.

*Theorem*: [*Extreme value theorem*] if a function $f$ is continuous on a closed, bounded set $D$, then it attains an absolute minimum and absolute maximum value on $D$.

*Example*: Let us illustrate the necessity of the hypotheses:

Summary: how to find the absolute max/min of a continuous function $f$ on a closed, bounded set $D$:

1. find the values of $f$ at all critical points
2. find the extreme (max/min) values on the boundary
3. take the overall largest/smallest value from the above

*Example*: find the absolute max. and min. of $f(x, y) = x^2 + 2xy + 3y^2$ on the set bounded by the triangle with vertices $(-1, 1)$, $(2, 1)$, and $(-1, -2)$.

# 10 Ch.15, Lecture 10 (Oct. 22)

### 15.8: Lagrange Multipliers

Problem: how do we find the maximum or minimum of a function when the variables are not free to take on any values, but rather satisy some *constraint*. For example, in the "aquarium example" above, the length, width, and height were not all allowed to vary freely: they had to satisfy the *constraint* that their product be equal to a fixed number (the volume, $V$). In that example, we could just solve the constraint to eliminate one variable. But this is not always possible.

Precise problem: find the maxima or minima of $f(x, y)$, subject to the constraint $g(x, y) = k$.

Consider the pictures:

*Theorem*: (Method of Lagrange Multipliers) Suppose $f$ and $g$ are differentiable functions. If $(a, b)$ is a local min. or max. of $f$ subject to the constraint $g(x, y) = k$ (meaning $f(x, y) \leq f(a, b)$ (resp. $f(x, y) \geq f(a, b)$) for all $(x, y)$ near $(a, b)$ satisfying $g(x, y) = k$), then

$$\nabla f(a, b) = \lambda \nabla g(a, b)$$

for some $\lambda \in \mathbb{R}$ (called a *Lagrange multiplier*).

*Sketch of a proof*:

*Example*: aquarium revisited:

*Example*: find the maximum and minimum of $x^2 + y^2 + z^2$ subject to $x^4 + y^4 + z^4 = 1$.

*Example*: (several constraints) find the maximum and minimum of $yz + xy$ subject to $xy = 1$ and $y^2 + z^2 = 1$.