Math 217: Vector Functions (Ch. 14)

1 Ch. 14, Lecture 1 (Sep. 17)

14.1: Vector Functions and Space Curves

Vector-valued function:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}.$$

Limits: *Definition*: the **limit** of $\mathbf{r}(t)$ as $t \to a$ is

$$\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$$

(if it exists).

Continuity: *Definition*: the vector function $\mathbf{r}(t)$ is **continuous** at a if

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a),$$

and is continuous on an interval I if it is continuous at each $a \in I$.

Definition: A **space curve** is the image of a continuous vector function defined on some interval I:

$$C = \{\mathbf{r}(t) \mid t \in I\} \subset \mathbb{R}^3.$$

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then **parametric equations** of C are

$$x = f(t), \quad y = g(t), \quad z = h(t).$$

Example: find a vector function which parameterizes the straight line through $\mathbf{r}_0$ in the direction $\mathbf{v}$.

Example: suppose the position of a particle at time t is given by $\mathbf{r}(t) = \langle \cos(t)\sin(t), \sin^2(t), \cos(t) \rangle$. Sketch the particle's trajectory.

Example: parameterize the curve of intersection of the parabolic cylinder $z = y^2$ and the plane $x + y = 1$.

14.2: Derivatives and Integrals of Vector Functions

Definition: the **derivative** of the vector function $\mathbf{r}(t)$ at a is

$$\mathbf{r}'(a) = \lim_{h \to 0} \frac{\mathbf{r}(a+h) - \mathbf{r}(a)}{h},$$

if it exists (in which case we say $\mathbf{r}$ is **differentiable** at a).

Clearly, if $\mathbf{r} = \langle f, g, h \rangle$, then $\mathbf{r}$ is differentiable at $a \Leftrightarrow f$, g, and h, are all differentiable at a, and, if so,

$$\mathbf{r}'(a) = \langle f'(a), g'(a), h'(a) \rangle.$$

Remark: We define higher derivatives $\mathbf{r}''$, $\mathbf{r}'''$, etc., in the same way.

Geometrically: if $\mathbf{r}(t)$ describes the space curve C, then $\mathbf{r}'(a)$ is a vector tangent to the curve C at $\mathbf{r}(a)$ (provided $\mathbf{r}'(a) \neq \mathbf{0}$).

Definition: Suppose a vector function $\mathbf{r}(t)$ parameterizes a space curve C. If $\mathbf{r}(t)$ is differentiable at $t = a$, and $\mathbf{r}'(a) \neq \mathbf{0}$, then

1. the **tangent line** to C at $\mathbf{r}(a)$ is given parametrically by $\mathbf{a}(a) + t\mathbf{r}'(a)$

2. the **unit tangent** to C at $\mathbf{r}(a)$ is the vector $\mathbf{T}(a) = \mathbf{r}'(a)/|\mathbf{r}'(a)|$.

Example: Find the unit tangent to the curve given by $\mathbf{r}(t) = R\langle \cos(t), \sin(t), 0 \rangle$.

Definition: We say the curve described by $\mathbf{r}(t)$, $a \leq t \leq b$ is **smooth** if $\mathbf{r}'(t)$ is continuous on (a, b), and $\mathbf{r}'(t) \neq \mathbf{0}$ for all $a < t < b$.

Example: A point on a rolling disk:

2 Ch. 14, Lecture 2 (Sep. 20)

Differentiation rules for vector functions

1. $[\mathbf{u}(t) + \mathbf{v}(t)]' = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $[c\mathbf{u}(t)]' = c\mathbf{u}'(t)$
3. $[f(t)\mathbf{u}(t)]' = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$
4. $[\mathbf{u}(t) \cdot \mathbf{v}(t)]' = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$
5. $[\mathbf{u}(t) \times \mathbf{v}(t)]' = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$
6. $[\mathbf{u}(f(t))]' = f'(t)\mathbf{u}'(f(t))$

Proof of (5), for example:

Example: Suppose the image of a vector function $\mathbf{r}(t)$ lies on the sphere of radius R centred at the origin. Show that the tangent vector to the curve is perpendicular to $\mathbf{r}(t)$.

Integrals: to integrate a vector function, we just integrate each component function.

Definition: If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is continuous for $a \leq t \leq b$, we define the **definite integral** of $\mathbf{r}$ over $[a, b]$ as

$$\int_a^b \mathbf{r}(t)dt := \left\langle \int_a^b f(t)dt, \quad \int_a^b g(t)dt, \quad \int_a^b h(t)dt \right\rangle.$$

By the fundamental theorem of calculus, if $\mathbf{R}'(t) = \mathbf{r}(t)$, then

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(b) - \mathbf{R}(a).$$

Example:

$$\int (e^t\hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + \ln(t)\hat{\mathbf{k}})dt =$$

14.3: Arc Length and Curvature

What is the length of the space curve C described by the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \leq t \leq b$.

Definition: So we define the **length** of C to be

$$L := \int_a^b |\mathbf{r}'(t)|dt = \int_a^b \left[(f'(t))^2 + (g'(t))^2 + (h'(t))^2\right]^{1/2} dt.$$

Definition: We also define the **arc length function** to be

$$s(t) := \int_a^t |\mathbf{r}'(u)|du$$

(so $s(a) = 0$ and $s(b) = L$).

Example: (2D) $\mathbf{r}(t) = \langle \cos(t), \sin(t)\rangle$, $0 \leq t \leq 2\pi$:

Example: $\mathbf{r}(t) = \langle \cos(\ln(t)), \sin(\ln(t)) \rangle$, $1 \leq t \leq e^{2\pi}$:

Remark: A given curve can have many different *parameterizations* (i.e. many different vector functions which describe it).

Example: $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$, $0 \leq t \leq 4\pi$:

One standard way to parameterize a curve is **by arc length**. Note that

$$\frac{d}{dt} s(t) = |\mathbf{r}'(t)|.$$

Example: Reparameterize the *helix* $\mathbf{r}(t) = \cos(t)\hat{\mathbf{i}} + \sin(t)\hat{\mathbf{j}} + t\hat{\mathbf{k}}$, $t \geq 0$ with respect to arc length.

Curvature

Let C be a smooth curve. So it is described by a vector function $\mathbf{r}(t)$, with $\mathbf{r}'(t) \neq 0$, and its unit tangent $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ is defined.

Definition: the **curvature** of C is

$$\kappa := \left|\frac{d\mathbf{T}}{ds}\right|$$

where s is arc length. To compute κ, note

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Example: Circle of radius R:

Example: Helix:

It is also sometimes useful to introduce the **unit normal** vector

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

and the **binormal** vector

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

Note $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are all of unit length, and orthogonal to each other.

3 Ch. 14, Lecture 3 (Sep. 22)

14.4: Motion in Space: Velocity and Acceleration

Suppose the position, at time t, of a particle moving in $\mathbb{R}^3$ is given by the vector function $\mathbf{r}(t)$. Then

- $\mathbf{v}(t) := \mathbf{r}'(t)$ is the particle's *velocity*
- $|\mathbf{v}(t)| = |\mathbf{r}'(t)|$ is the particle's *speed*
- $\mathbf{a}(t) := \mathbf{v}'(t) = \mathbf{r}''(t)$ is the particle's *acceleration*

Example: Starting at the origin with velocity $\hat{\mathbf{k}}$, an object's acceleration is $e^t\hat{\mathbf{i}} + e^{-t}\hat{\mathbf{j}}$. Find its position as a function of time.

Example: Find the force acting on an object with mass m moving around a circle of radius R in the xy-plane at constant angular speed ω.