Math 217: Vectors and the Geometry of Space (Ch. 13)

1 Ch. 13, Lecture 1 (Sep. 10)

13.1: 3D Coordinate Systems

$\mathbb{R} = \{x \mid -\infty < x < \infty\}$

$\mathbb{R}^2 = \{(x, y) \mid x, y, \in \mathbb{R}\}$

$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

Coordinate planes:

The xy-**plane** in $\mathbb{R}^3$ is $\{(x, y, 0) \mid x, y, \in \mathbb{R}\} = \{z = 0\}$, etc.

Example: What set is described by

1. $y = 2$?
2. $y = x$?

Distance formula: the distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example: the sphere of radius r and centre (a, b, c) consists of all points in $\mathbb{R}^3$ whose distance to the point (a, b, c) is r:

$$\{(x, y, z) \mid \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r\}.$$

So we can write the equation of this surface as $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$.

13.2 Vectors

Informally, a **vector** is a quantity with both a magnitude and a direction (think of a velocity, or a force).
We can represent a vector as an arrow in $\mathbb{R}^3$ (or $\mathbb{R}^2$) where

- direction of arrow ↔ direction of vector
- length of arrow ↔ magnitude of vector

If we put the base of the arrow at the origin, we can identify a vector with the coordinates of its endpoint. So
a vector in $\mathbb{R}^3$ (or $\mathbb{R}^2$) $\leftrightarrow$ an ordered triple (or pair): $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

The **magnitude** of the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

(or $|\langle a_1, a_2 \rangle| = \sqrt{a_1^2 + a_2^2}$ in 2D).

Remark: The **zero vector** $\mathbf{0} = \langle 0, 0, 0 \rangle$ is special: it has magnitude zero, and no direction.

Vector addition: geometrically, we add vectors by placing them head to tail. Analytically, this means that if $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, we have

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle.$$

Scalar multiplication of vectors: If c is a "scalar" (i.e. a real number) and $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, we define

$$c\mathbf{a} = \langle ca_1, ca_2, ca_3\rangle.$$

Geometrically, multiplying $\mathbf{a}$ by c

- stretches it by $|c|$
- reflects it in the origin in $c < 0$.

Note: $|c\mathbf{a}| = |c||\mathbf{a}|$.

Definition: We say two vectors $\mathbf{a}$ and $\mathbf{b}$ are **parallel** if there is $c \in \mathbb{R}$ such that $\mathbf{a} = c\mathbf{b}$.

The difference of two vectors: $\mathbf{a} - \mathbf{b} := \mathbf{a} + (-1)\mathbf{b}$.

Analytically: $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$.

Geometrically:

Properties of vector arithmetic: see list on p. 838 of text.
Upshot: vector addition and scalar multiplication behaves just like that for usual numbers.

Standard basis vectors:

$$\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle, \quad \hat{\mathbf{j}} = \langle 0, 1, 0 \rangle, \quad \hat{\mathbf{k}} = \langle 0, 0, 1 \rangle.$$

Any vector can be expressed in terms of the standard basis vectors:
$\langle a_1, a_2, a_3 \rangle = a_i\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$,
$\langle 2, -4, 5 \rangle = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, etc.

Note: $|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1$.

Definition: A vector with magnitude 1 is called a **unit vector**.

Given any vector $\mathbf{a} \neq \mathbf{0}$, there is a unit vector with the same direction as $\mathbf{a}$:

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

2 Ch. 13, Lecture 2 (Sep. 13)

13.3 Dot Product

Definition: The **dot product** of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is the number

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Properties:

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{v}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{v}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

Geometric picture:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

Proof of this:

Note: if $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors,

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Definition: We say two non-zero vectors are **orthogonal** (or **perpendicular**) if the angle between them is $\pi/2$.

So we see $\mathbf{a}$ and $\mathbf{b}$ are orthogonal $\Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$.

Note: if $\mathbf{a}$ and $\mathbf{b}$ are parallel, say $\mathbf{b} = c\mathbf{a}$, then $\cos(\theta) = c\mathbf{a} \cdot \mathbf{a}/(|\mathbf{a}||c||\mathbf{a}|) = \pm 1$. So $\theta = 0$ or $\theta = \pi$.

Projections

Definition: The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ is

$$comp_{\mathbf{a}}\mathbf{b} := \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}.$$

The **projection** of $\mathbf{b}$ onto $\mathbf{a}$ is

$$proj_{\mathbf{a}}\mathbf{b} := (comp_{\mathbf{a}}\mathbf{b})\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{a}}{|\mathbf{a}|^2}.$$

Example: Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. Find $proj_{\mathbf{i}}\mathbf{a}$.

13.3 Cross Product

Definition: The **cross product** (or **vector product**) of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is the vector

$$\mathbf{a} \times \mathbf{b} := \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

Determinant notation for remembering this formula:

Geometric picture of the cross product. Check that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$:

We also have $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$.

In particular, $\mathbf{a}$ and $\mathbf{b}$ are parallel $\Leftrightarrow$ $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

A special case of this fact: $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

A geometric interpretation of the cross product:

Example: Find the area of the triangle shown:

Properties of the cross-product:

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{v}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{v}$
4. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{v}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{v}$
5. $\mathbf{a} \times (\mathbf{b} \times \mathbf{v}) = (\mathbf{a} \cdot \mathbf{v})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{v}$

Some remarks:

- (1) is important – cross products do not “commute”!
- The quantity in (4) is the **triple product**. Geometric interpretation:
- (5) is not equal to $(\mathbf{a} \times \mathbf{b}) \times \mathbf{v}$ in general – the cross product is not “associative”!

3 Ch. 13, Lecture 3 (Sep. 15)

13.5 Equations of Lines and Planes

The line L through a point $\mathbf{r}_0 \in \mathbb{R}^3$ in the direction $\mathbf{v}$ is

$$L = \{\mathbf{r}_0 + t\mathbf{v} \in \mathbb{R}^3 \mid t \in \mathbb{R}\}.$$

That is, L is described by the *parametric* equation $\mathbf{r}_0 + t\mathbf{v}$ as the *parameter* t runs over all values $-\infty < t < \infty$.

This is equivalent to the three scalar parametric equations

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

where $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, and $\mathbf{v} = \langle a, b, c \rangle$.

Example: Find a vector equation for the line passing through the points $\mathbf{r}_0$ and $\mathbf{r}_1$.

Another way to describe L is to eliminate the parameter t from the scalar parametric equations above, to get the so-called **symmetric equations** for L:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

This makes sense if a, b, and c are not all 0 (if, for example, $a = 0$, we have $x = x_0$ and $(y - y_0)/b = (z - z_0)/c$).

Example: Find parametric and symmetric equations for the line through $(1, 0, 0)$ and $(1, 1, 1)$.

Planes: a plane in $\mathbb{R}^3$ is determined by a point, $\mathbf{r}_0$, on it, and a vector, $\mathbf{n}$, orthogonal to it. The vector $\mathbf{n}$ is called a **normal vector** to the plane.

Thus a point $\mathbf{r} = \langle x, y, z\rangle$ is on the plane if and only if

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

This is an equation of the plane.

If $\mathbf{n} = \langle a, b, c\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$, the corresponding scalar equation is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Another way to write this is

$$ax + by + cz = d$$

(here we have $d = ax_0 + by_0 + cz_0$).

Example: Find an equation for the plane passing through $(1, 2, -2)$, $(1, -1, 1)$, and $(0, 0, 1)$. Try and sketch this plane.

Example: Find a formula for the distance from the point (x_0, y_0, z_0) to the plane $ax + by + cz = d$.

13.6 Cylinders and Quadric Surfaces

Definition: A **cylinder** is a surface consisting of all lines parallel to a given line and passing through a given plane curve.

Example: Sketch the graph of $x^2 + y^2/4 = 1$.

Some other surfaces:

Example: Sketch the graph of $z = y^2$.

Example: Sketch the graph of $y = 4x^2 + z^2$.

Example: Sketch the graph of $z = y^2 - x^2$.

13.7 Cylindrical and Spherical Coordinates

...we'll skip this for now, and come back to it later.