Nonlinear Systems

- we now consider 2D systems of autonomous but nonlinear ODE: $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$;

$$(*) \quad \boxed{\vec{x}' = \vec{f}(\vec{x})} \qquad \text{where} \quad \vec{f}(\vec{x}) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

- since they are nonlinear, we stand no chance of being able to solve them explicitly. Nevertheless, we can do some analysis which will give us a good idea of how solutions behave

Critical Points: the one class of solutions which are easy to find are the constant ones:

$$\vec{x}(t) = \vec{x}_0 \quad (\text{independent of } t)$$

for which $\vec{x}' = \vec{0}$ and hence $\vec{f}(\vec{x}(t)) = \vec{f}(\vec{x}_0) = \vec{0}$

- a point $\vec{x}_0$ (in the plane) for which $\vec{f}(\vec{x}_0) = \vec{0}$ is called a critical point of $(*)$

- the corresponding constant solution $\vec{x}(t) = \vec{x}_0$ is called an equilibrium solution

<u>Example:</u> for the "competing species" system
(here $x(t)$ = # of species 1, $y(t)$ = # of species 2)

natural growth rate

reduction in growth rate due to competition

$$\begin{cases} \dfrac{dx}{dt} = x(a_1 - b_1 x - c_1 y) \\[2ex] \dfrac{dy}{dt} = \underbrace{y(a_2 - b_2 y}_{} - c_2 x) \end{cases}$$

reduction in growth rate due to environmental limits

this part is a 'logistic equation' (1^{st}-order autonomous ODE)

with $a_1 = b_1 = c_1 = 1$ $\quad a_2 = 3/4$, $b_2 = 1$, $c_2 = 1/2$,
find all the critical points:

$$x(1-x-y) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x+y=1$$
$$y(\tfrac{3}{4} - y - \tfrac{1}{2}x) = 0 \quad \Rightarrow \quad y = 0 \text{ or } \tfrac{1}{2}x + y = 3/4$$

& so we have four solutions: $x=0, y=0$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

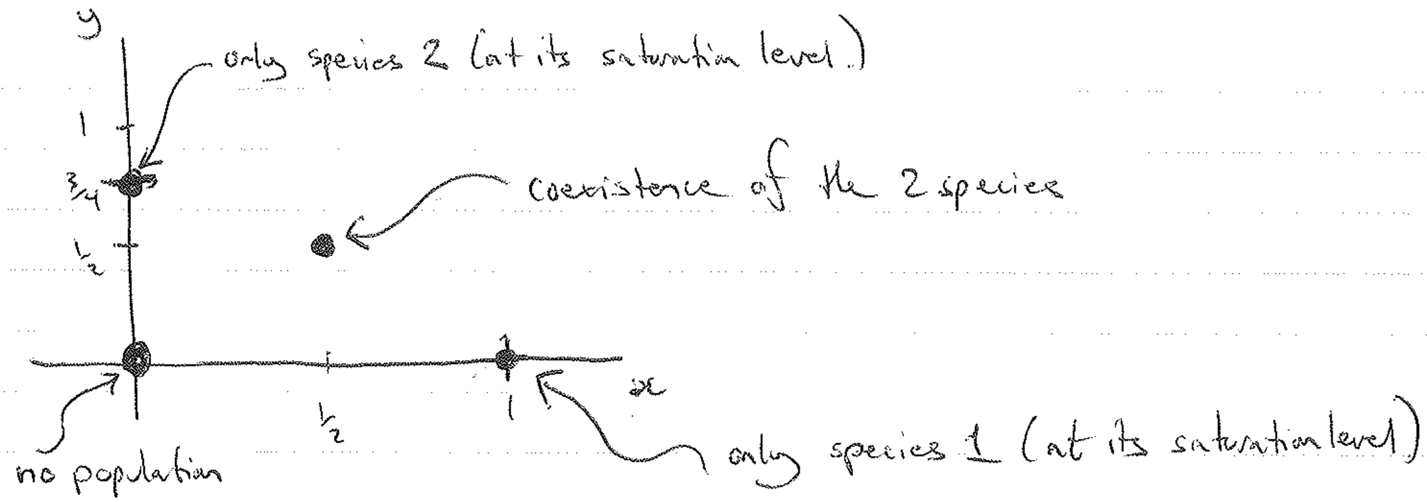
$x=0, y=3/4$: $\begin{bmatrix} 0 \\ 3/4 \end{bmatrix}$

$x=1, y=0$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\left\{\begin{matrix} x+y=1 \\ \tfrac{1}{2}x + y = 3/4 \end{matrix}\right\} \Rightarrow x = 1/2, y = 1/2 : \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

- if we plot these equilibrium solutions as <u>trajectories</u> (solutions $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ plotted in the xy-plane)

in the <u>phase plane</u> (the xy-plane), we have:



- ultimately, we wish to fill up the phase plane with typical trajectories, but the next step is more modest: understand behaviour of solutions very close to each critical point

<u>Linearization</u>: of (*), and write
if $\vec{x}_0$ is a critical point $(\vec{f}(\vec{x}_0) = \vec{0})$ and $\vec{x}(t)$ is a solution close to $\vec{x}_0$, we write

$$\vec{x}(t) = \vec{x}_0 + \underbrace{\vec{u}(t)}_{\text{small!}}, \qquad \vec{u}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \text{ small!}, \qquad \vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

plug into the ODE system (*), and use the linear approximation:

$$\vec{u}' = \vec{x}' = \vec{f}(\vec{x}) = \vec{f}(\vec{x}_0 + \vec{u})$$

$$= \begin{bmatrix} f(x_0+u, y_0+v) \\ g(x_0+u, y_0+v) \end{bmatrix} \approx \begin{bmatrix} f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)u + \frac{\partial f}{\partial y}(x_0,y_0)v \\ g(x_0,y_0) + \frac{\partial g}{\partial x}(x_0,y_0)u + \frac{\partial g}{\partial y}(x_0,y_0)v \end{bmatrix}$$

$$= \cancelto{\vec{0}}{\vec{f}(\vec{x}_0)} + D\vec{f}(\vec{x}_0)\vec{u}$$

where $D\vec{f}(\vec{x}_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix}$ is the <u>Jacobian matrix</u> of $\vec{f}$ at $\vec{x}_0$

- thus, approximately (as long as $\vec{v}(t)$ remains small), the deviation $\vec{v}(t)$ (from the critical point $\vec{x_0}$) satisfies the linear system

$$\boxed{\vec{v}\,' = J\vec{v}}, \qquad J = D\vec{f}(\vec{x_0})$$

called the <u>linearization</u> of (*) at the critical point $\vec{x_0}$.

- since J is a constant matrix, we know how to solve the linearization explicitly by considering eigenvalues and eigenvectors. In this way, we get a picture of how solutions of (*) behave nearby each critical point.

<u>Example:</u> for each of the critical points $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\3/4\end{bmatrix}, \begin{bmatrix}1/2\\1/2\end{bmatrix}$ of the 'competing species' system

$$\left\{\begin{aligned} \frac{dx}{dt} &= x(1-x-y) \\ \frac{dy}{dt} &= y\left(\tfrac{3}{4}-y-\tfrac{1}{2}x\right) \end{aligned}\right\}$$

find the linearization, determine the type and stability of the critical point, and sketch the phase portrait near the critical point.

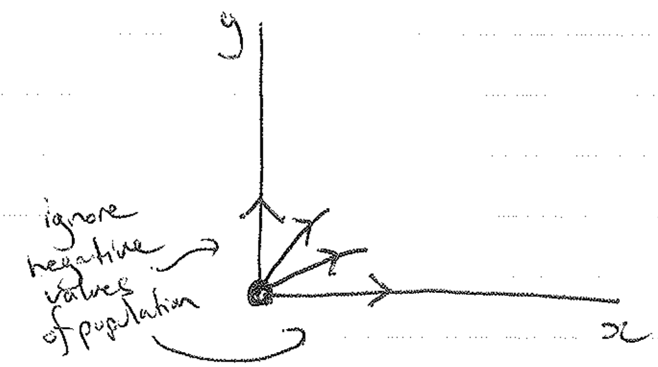
- we need the Jacobian matrix. With $\vec{f} = \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} x - x^2 - xy \\ \frac{3}{4}y - y^2 - \frac{1}{2}xy \end{bmatrix}$, we have

$$D\vec{f}(\vec{x}) = \begin{bmatrix} 1-2x-y & -x \\ -\frac{1}{2}y & \frac{3}{4}-2y-\frac{1}{2}x \end{bmatrix}$$

- $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$: $J = D\vec{f}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 3/4 \end{bmatrix}$ has eigenvalues

  1 (eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$) and $3/4$ (eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

  so this is a <u>source</u> (or <u>unstable node</u>), it is <u>unstable</u> and the trajectories near $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ look like

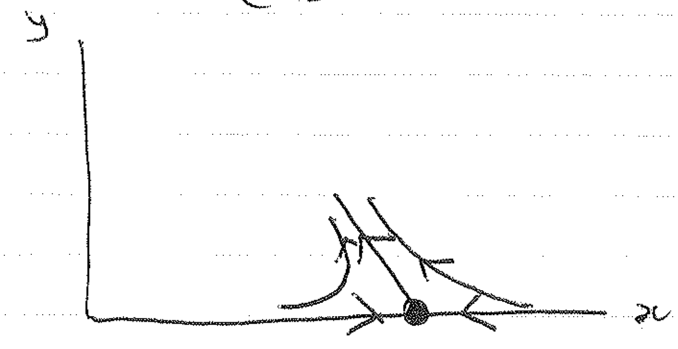


- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$: $J = D\vec{f}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 & -1 \\ 0 & 1/4 \end{bmatrix}$ has eigenvalues $-1, \frac{1}{4}$

  - eigenvectors: $\lambda = -1$: $\begin{bmatrix} 0 & -1 \\ 0 & 5/4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

    $\lambda = \frac{1}{4}$: $\begin{bmatrix} -5/4 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

so this is a <u>saddle point</u>, it is <u>unstable</u>, and the trajectories near $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ look like:

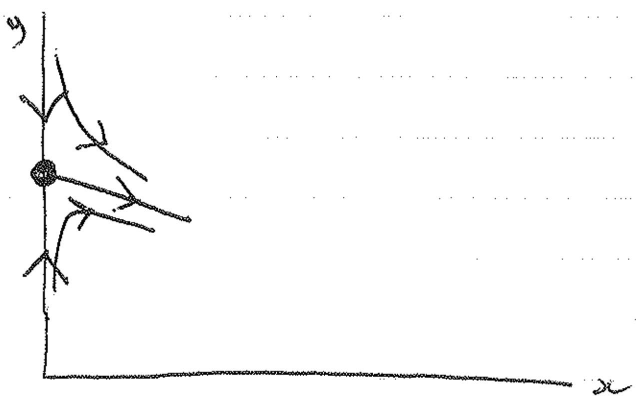


- $\begin{bmatrix} 0 \\ 3/4 \end{bmatrix}$: $J = D\vec{f}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{4} & 0 \\ -3/8 & -3/4 \end{bmatrix}$ has eigenvalues $-3/4$, $1/4$

  - eigenvectors: $\lambda = -3/4$: $\begin{bmatrix} 1 & 0 \\ -3/8 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

    $\lambda = 1/4$: $\begin{bmatrix} 0 & 0 \\ -3/8 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 8 \\ -3 \end{bmatrix}$

so this is a <u>saddle point</u>, it is <u>unstable</u>, and the trajectories near $\begin{pmatrix} 0 \\ 3/4 \end{pmatrix}$ look like



- $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$: $J = D\vec{f}\left(\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}\right) = \begin{bmatrix} -1/2 & -1/2 \\ -1/4 & -1/2 \end{bmatrix}$

  - eigenvalues: $0 = \det \begin{bmatrix} -\frac{1}{2}-\lambda & -1/2 \\ -1/4 & -1/2-\lambda \end{bmatrix} = \left(\lambda + \frac{1}{2}\right)^2 - \frac{1}{8} = \lambda^2 + \lambda + \frac{1}{8}$

$$\Rightarrow \lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{8}} = -\frac{1}{2} \pm \sqrt{\frac{1}{8}}$$

  - both eigenvalues are negative, so this is a <u>sink</u> (or <u>stable node</u>), which is <u>stable</u>

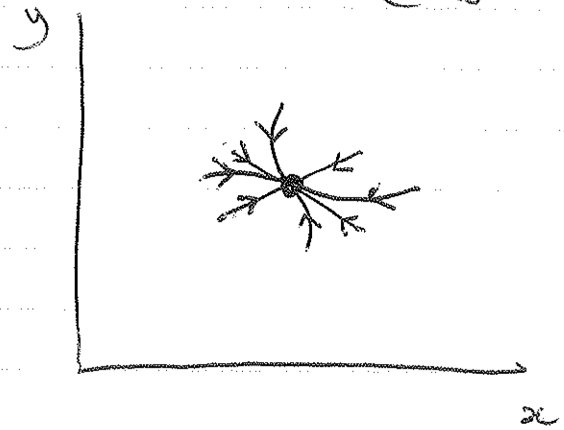
(in fact "<u>asymptotically stable</u>": nearby trajectories approach the critical point as $t \to \infty$).

- to get a reasonable picture, let's find eigenvectors:

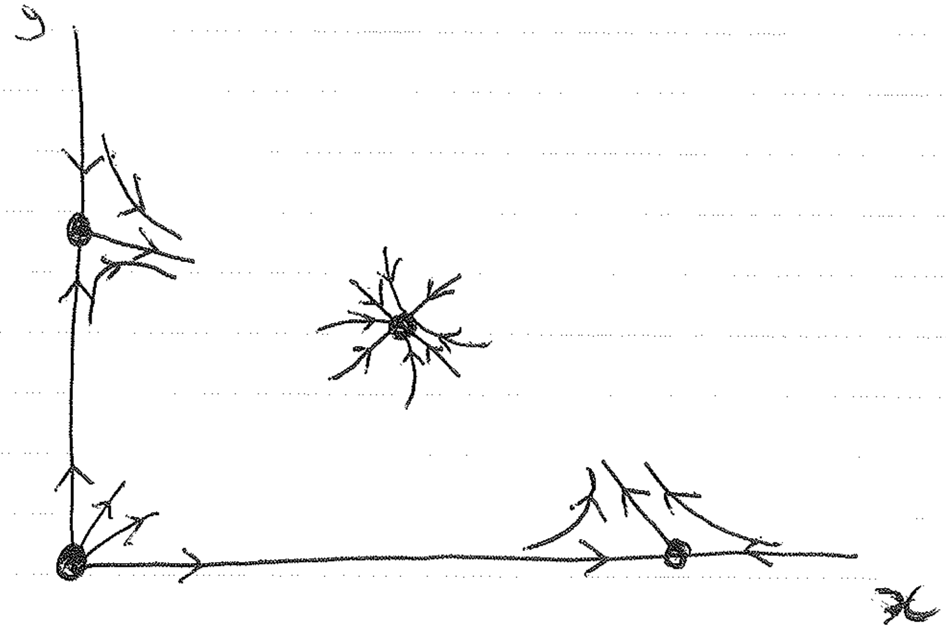
$$\lambda = -\tfrac{1}{2} + \tfrac{1}{\sqrt{8}} : \begin{bmatrix} -\frac{1}{\sqrt{8}} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{\sqrt{8}} \end{bmatrix} \begin{bmatrix} \sqrt{8} \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{8} \\ -2 \end{bmatrix}$$

$$\lambda = -\tfrac{1}{2} - \tfrac{1}{\sqrt{8}} : \begin{bmatrix} \frac{1}{\sqrt{8}} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{\sqrt{8}} \end{bmatrix} \begin{bmatrix} \sqrt{8} \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{8} \\ 2 \end{bmatrix}$$

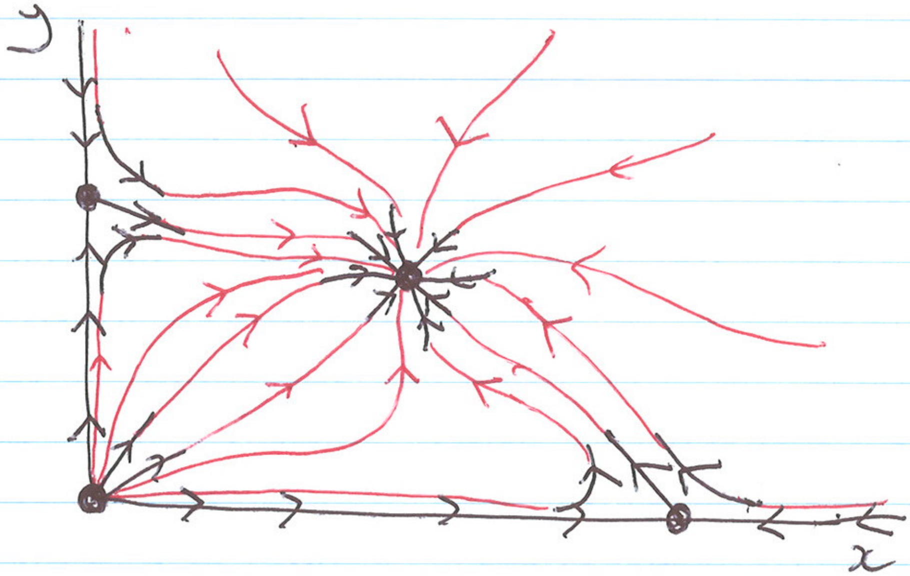
so the trajectories near $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ look like



Now we can put our 4 "local" pictures together on the phase plane:



... and then make our best effort to consistently fill the phase plane with typical trajectories:



Remarks:

1. of course this last part is somewhat nonrigorous and speculative! Nevertheless, we get a qualitatively correct picture of how solutions behave.

2. our conclusion: for initial conditions with positive populations of both species ($x(0)>0$, $y(0)>0$), the solution will ultimately tend toward the 'coexistence' critical point $x=1/2$, $y=1/2$

3. notice that if $x(0)=0$ or $y(0)=0$ (initial conditions on the axes), the solution stays on the axis and is governed by a 1^{st}-order autonomous ODE:

$$\frac{dx}{dt} = x(1-x) \quad (\text{if } y=0) \qquad \text{or} \qquad \frac{dy}{dt} = y(3/4-y) \quad (\text{if } x=0)$$

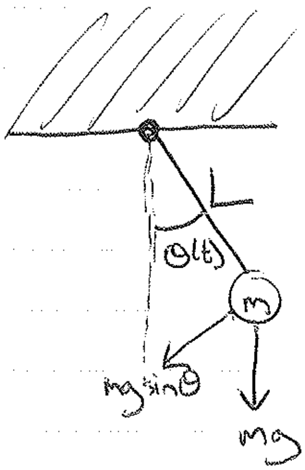
4. one can get a slightly more accurate picture by considering the nullclines, the lines $x+y=1$ and $\frac{1}{2}x+y=3/4$ (other than the axes) where

either $\frac{dx}{dt}=0$ (so trajectories are vertical there ↓ or ↑)

or $\frac{dy}{dt}=0$ (" " " horizontal " → or ←)

but we will not do so here.

<u>Example</u>: the damped, nonlinear pendulum



$$\text{Newton: } \quad mL\theta'' = -mg\sin\theta - \underbrace{mg\overset{\text{damping constant}}{c}\,\theta'}_{\text{damping term}}$$

nonlinear!!

$$\Rightarrow \quad \theta'' = -\frac{g}{L}\sin\theta - c\theta'$$

- 1^{st} re-write this 2^{nd} order ODE as a 1^{st}-order system:

$$\vec{x}(t) = \begin{bmatrix}\theta(t)\\ \theta'(t)\end{bmatrix} \Rightarrow \vec{x}' = \begin{bmatrix}\theta'\\ \theta''\end{bmatrix} = \begin{bmatrix}\theta'\\ -\frac{g}{L}\sin\theta - c\theta'\end{bmatrix} =: \vec{f}(\vec{x})$$

- the critical points satisfy

$$\theta' = 0$$
$$-g/_L\sin\theta - c\theta' = 0 \Rightarrow \sin\theta = 0$$

and so are $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pm\pi\\0\end{bmatrix}, \begin{bmatrix}\pm 2\pi\\0\end{bmatrix}, \begin{bmatrix}\pm 3\pi\\0\end{bmatrix}$, etc.

"down"  "up"  down up

- the Jacobian matrix is

$$D\vec{f}(\vec{x}) = \begin{bmatrix} \frac{\partial}{\partial\theta}(\theta') & \frac{\partial}{\partial\theta'}(\theta') \\ \frac{\partial}{\partial\theta}\left(-\frac{g}{L}\sin\theta - c\theta'\right) & \frac{\partial}{\partial\theta'}\left(-\frac{g}{L}\sin\theta - c\theta'\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos\theta & -c \end{bmatrix}$$

- linearization at critical point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ("down"):

$$J = D\vec{f}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -g/L & -c \end{bmatrix}$$

- eigenvalues: $0 = \det\begin{pmatrix} -\lambda & 1 \\ -g/L & -c-\lambda \end{pmatrix} = \lambda^2 + c\lambda + g/L$

$$\Rightarrow \quad \lambda = -\frac{c}{2} \pm \sqrt{c^2/4 - g/L}$$

<u>Case 1</u> (overdamped): $c^2/4 > g/L \Rightarrow$ 2 negative eigenvalues $\Rightarrow$ <u>sink</u> (<u>stable node</u>) which is <u>(asymptotically) stable</u>

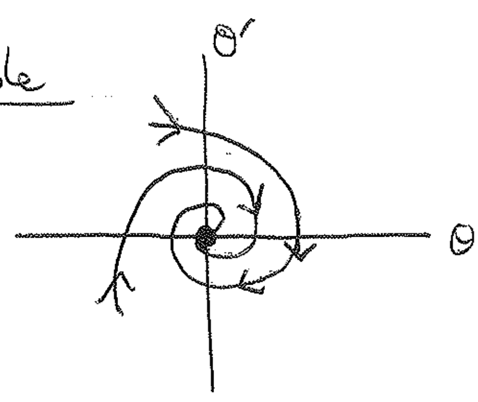
proceed with this case

<u>Case 2</u> (underdamped): $c^2/4 < g/L \Rightarrow \lambda = -\frac{c}{2} \pm \sqrt{\frac{g}{L} - \frac{c^2}{4}}\, i$

complex eigenvalues with negative real part

$\Rightarrow$ <u>spiral sink</u>, <u>(asymptotically) stable</u>

- note $J\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -g/L \end{bmatrix}$ $\downarrow$, so direction of spiral is



- at critical point $\begin{bmatrix}\pi\\0\end{bmatrix}$ ("up")

$$J = D\vec{f}\left(\begin{bmatrix}\pi\\0\end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ g/L & -c \end{bmatrix}$$ whose eigenvalues are given by

$$0 = \det\begin{bmatrix} -\lambda & 1 \\ g/L & -\lambda-c \end{bmatrix} = \lambda^2 + c\lambda - g/L$$

$$\Rightarrow \lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + g/L}\ ,$$ 2 eigenvalues of opposite sign

$\Rightarrow$ a <u>saddle point</u>, which is <u>unstable</u>

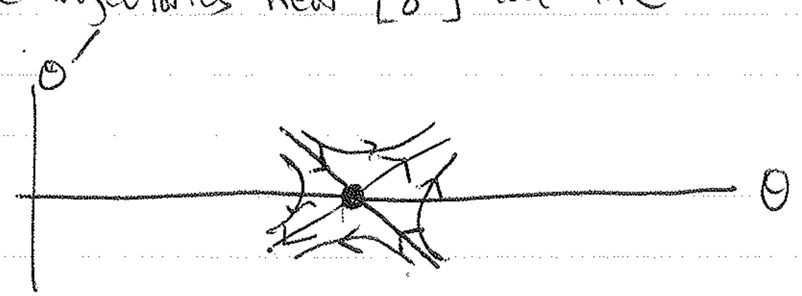
(of course a pendulum pointing straight up is unstable!)

- eigenvectors:

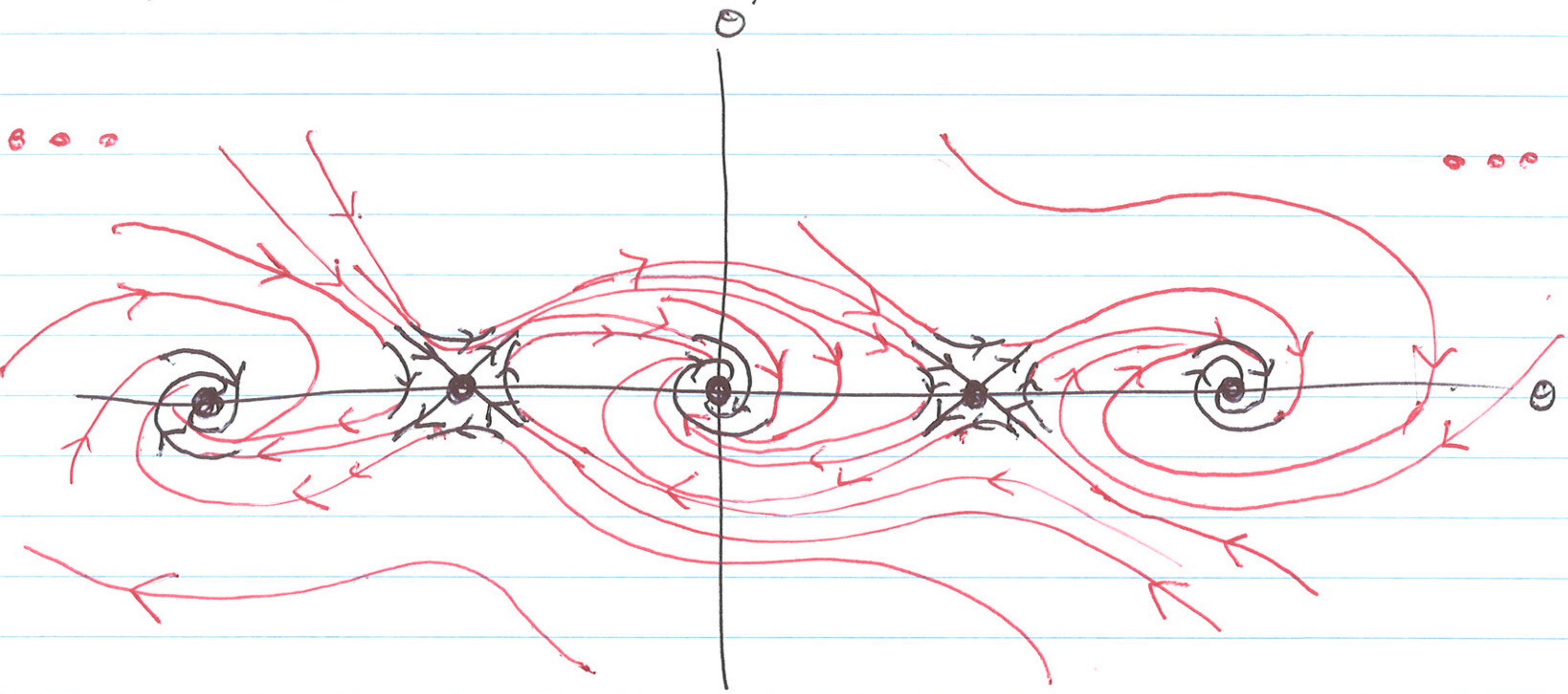
$$\lambda = -\frac{c}{2} + \sqrt{\cdots} : \quad \begin{bmatrix} \frac{c}{2} - \sqrt{\cdots} & 1 \\ g/L & -\frac{c}{2} - \sqrt{\ } \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{c}{2} + \sqrt{\ } \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda = -\frac{c}{2} - \sqrt{\cdots} : \quad \begin{bmatrix} \frac{c}{2} + \sqrt{\ } & 1 \\ g/L & -\frac{c}{2} + \sqrt{\ } \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{c}{2} - \sqrt{\ } \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- so the trajectories near $\begin{bmatrix}\pi\\0\end{bmatrix}$ look like



- finally, we attempt to piece together the full phase portrait:



Remark: notice there are a few very special trajectories (the incoming ones into the saddle points (the "ups")) which have just the right initial velocity from a given starting position to exactly approach the "upward"

position as $t \to \infty$. But this is very rare!

A "generic" choice of initial position and velocity will result in the pendulum settling down (perhaps after a few rotations) toward the "down"

position.