Nonlinear Systems

- We now consider 2D systems of autonomous
  but nonlinear ODE: \( \dot{x}(t) = f(x(t)) \)

\[ (*) \quad \dot{x} = f(x) \quad \text{where} \quad f(x) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} \]

- Since they are nonlinear, we stand no chance of being able to solve them explicitly. Nevertheless, we can do some analysis which will give us a good idea of how solutions behave.

Critical Points: the one class of solutions which are easy to find are the constant ones:

\[ \dot{x}(t) = \dot{x}_0 \quad \text{(independent of } t) \]

For which \( \dot{x} = 0 \) and hence \( f(x(t)) = f(x_0) = 0 \)

- A point \( x_0 \) (in the plane) for which \( f(x_0) = 0 \) is called a critical point of \((*)\)

- The corresponding constant solution \( x(t) = x_0 \) is called an equilibrium solution
Example 2 for the "competing species" system

(Here \( x(t) = \# \) of species 1, \( y(t) = \# \) of species 2 )

\[
\begin{align*}
\frac{dx}{dt} &= x(a_1 - b_1x - c_1y) \\
\frac{dy}{dt} &= y(a_2 - b_2y - c_2x)
\end{align*}
\]

This part is a 'logistic equation' (1\textsuperscript{st} order autonomous ODE)

with \( a_1 = b_1 = c_1 = 1, a_2 = 3/4, b_2 = 1, c_2 = 1/2 \)

find all the critical points :

\[
\begin{align*}
x(1-x-y) &= 0 \Rightarrow x = 0 \text{ or } x+y=1 \\
y(3/4 - y - 1/2x) &= 0 \Rightarrow y = 0 \text{ or } 1/2x + y = 3/4
\end{align*}
\]

so we have four solutions: \( x=0, y=0 : \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \)
\( x=0, y=3/4 : \left[ \begin{array}{c} 0 \\ 3/4 \end{array} \right] \)
\( x=1, y=0 : \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \)
\( x=1/2, y=3/4 \)

if we plot these equilibrium solutions as trajectories (solutions \( x(t), y(t) \) plotted in the \( xy \)-plane), in the \( xy \)-plane (the \( xy \)-plane), we have:
ultimately, we wish to fill up the phase plane with typical trajectories, but the next step is more modest: understand behaviour of solutions very close to each critical point.

**Linearization:** if \( \dot{\mathbf{x}}_0 \) is a critical point \( (\dot{f}(\mathbf{x}_0) = \mathbf{0}) \) of \((\mathbf{f})\), and \( \mathbf{x}(t) \) is a solution close to \( \mathbf{x}_0 \), we write

\[
\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{\hat{u}}(t),
\]

where \( \mathbf{\hat{u}}(t) \) is small.

\[
\mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}
\]

plug into the ODE system \((\mathbf{f})\), and use the linear approximation:

\[
\begin{align*}
\dot{\mathbf{\hat{u}}} &= \mathbf{\hat{u}}' = \mathbf{f}(\mathbf{\hat{x}}) = \mathbf{f}(\mathbf{x}_0 + \mathbf{\hat{u}}) \\
&= \begin{bmatrix} f(x_0 + u, y_0 + v) \\ g(x_0 + u, y_0 + v) \end{bmatrix} \\
&\approx \begin{bmatrix} f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) u + \frac{\partial f}{\partial y}(x_0, y_0) v \\ g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0) u + \frac{\partial g}{\partial y}(x_0, y_0) v \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{f}(\mathbf{x}_0) \\ \mathbf{g}(\mathbf{x}_0) \end{bmatrix} + \mathbf{Df}(\mathbf{x}_0) \mathbf{\hat{u}}
\end{align*}
\]

where \( \mathbf{Df}(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix} \) is the **Jacobian Matrix** of \( \mathbf{f} \) at \( \mathbf{x}_0 \).
Thus, approximately (as long as \( \tilde{u}(t) \) remains small), the deviation \( \tilde{u}(t) \) (from the critical point \( \tilde{x}_0 \)) satisfies the linear system

\[
\dot{\tilde{u}} = J \tilde{u}, \quad J = \frac{\partial f}{\partial x}(x_0)
\]

called the linearization of (8) at the critical point \( \tilde{x}_0 \).

Since \( J \) is a constant matrix, we know how to solve the linearization explicitly by considering eigenvalues and eigenvectors. In this way, we get a picture of how solutions behave nearby each critical point.

**Examples:** For each of the critical points \( (0), (0, \frac{1}{2}), (\frac{1}{2}, 0) \) of the competitive species' system

\[
\begin{cases}
\frac{dx}{dt} = x(1-x-y) \\
\frac{dy}{dt} = y(3y-x-\frac{1}{2}x)
\end{cases}
\]

find the linearization, determine the type and stability of the critical point, and sketch the phase portrait near the critical point.

We need the Jacobian matrix. With \( \nabla f = \left[ \begin{array}{c} f_x \\ f_y \end{array} \right] = \left[ \begin{array}{c} x - x^2 - xy \\ 3y^2 - y^2 - \frac{1}{2}x \end{array} \right] \)

we have

\[
\frac{\partial f}{\partial x}(x_0) = \left[ \begin{array}{ccc} 1 - 2x & -x \\ -\frac{1}{2} & 3y^2 - y^2 - \frac{1}{2}x \end{array} \right]
\]
\[ J = D\tilde{f}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 3/4 \end{bmatrix} \] has eigenvalues 1 (eigenvector \([1]\)) and 3/4 (eigenvector \([0]\)).

So this is a source (or unstable node), it is unstable, and the trajectories near \([0]\) look like:

\[ \begin{aligned}
&\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{has eigenvalues } -1, \frac{1}{4} \\
\text{eigenvalues: } x = -1: \begin{bmatrix} 0 & -1 \\ 0 & 3/4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\text{and } x = 3/4: \begin{bmatrix} -3/4 & -1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 \\ -5 \end{bmatrix}
\end{aligned} \]

So this is a saddle point, it is unstable, and the trajectories near \([10]\) look like:
\[ J = Df \left( \begin{bmatrix} 0 \\ 3/4 \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{4} & 0 \\ -3/8 & -3/4 \end{bmatrix} \text{ has eigenvalues } \lambda = \frac{-3}{4}, \frac{1}{4} \]

- eigenvectors:
  \[ \lambda = \frac{-3}{4} : \begin{bmatrix} 1 & 0 \\ -3/8 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

  \[ \lambda = \frac{1}{4} : \begin{bmatrix} 0 & 0 \\ -3/8 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -3 \end{bmatrix} \]

So this is a saddle point, it is unstable, and the trajectories near \( \begin{bmatrix} 0 \\ 3/4 \end{bmatrix} \) look like ...

\[ \begin{bmatrix} 0 \\ 3/4 \end{bmatrix} \]

\[ J = Df \left( \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \right) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \]

- eigenvalues:
  \[ \det \begin{bmatrix} -\frac{1}{2} - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} - \lambda \end{bmatrix} = \left( \lambda + \frac{1}{2} \right)^2 - \frac{1}{8} = \lambda^2 + \lambda + \frac{1}{8} \]

  \[ \Rightarrow \lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{8}} = -\frac{1}{2} \pm \frac{\sqrt{1}}{8} \]

- both eigenvalues are negative, so this is a sink (or stable node), which is stable (in fact, "asymptotically stable"; nearby trajectories approach the critical point as \( t \to \infty \)).
to get a reasonable picture, let's find eigenvectors:

\[ \lambda = -\frac{1}{2} + \frac{1}{\sqrt{8}} : \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} \sqrt{8} \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{8} \\ -2 \end{bmatrix} \]

\[ \lambda = -\frac{1}{2} - \frac{1}{\sqrt{8}} : \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} \sqrt{8} \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{8} \\ 2 \end{bmatrix} \]

so the trajectories near \( \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \) look like

\[ \begin{array}{c}
\begin{array}{c}
\text{y} \\
\text{x}
\end{array}
\end{array} \]

Now we can put our 4 "local" pictures together on the phase plane.

\[ \begin{array}{c}
\begin{array}{c}
\text{y} \\
\text{x}
\end{array}
\end{array} \]
... and then make our best effort to consistently fill the phase plane with typical trajectories:

\[ y \]
\[ x \]

**Remarks:**

1. Of course this last part is somewhat non-rigorous and speculative! Nonetheless, we get a qualitatively correct picture of how solutions behave.

2. Our conclusion: for initial conditions with positive populations of both species \((x(0) > 0, y(0) > 0)\), the solution will ultimately tend toward the 'coexistence' critical point \(x = \frac{1}{2}, y = \frac{1}{2}\).

3. Notice that if \(x(0) = 0\) or \(y(0) = 0\) (initial conditions on the axes), the solution stays on the axis and is governed by a 1st-order autonomous ODE:
   \[
   \frac{dx}{dt} = x(1-x) \quad \text{(if } y=0) \quad \text{or} \quad \frac{dy}{dt} = y\left(\frac{3}{4} - y\right) \quad \text{(if } x=0)\]
4. One can get a slightly more accurate picture by considering the **nullclines**, the lines $x+y=1$ and $\frac{1}{2}x+y=\frac{3}{4}$ (other than the axes) where either $\frac{dx}{dt}=0$ (so trajectories are vertical there & or↑) or $\frac{dy}{dt}=0$ (↑ or horizontal u → or ↓).

But we will not do so here.

**Examples** the damped, nonlinear pendulum

![Diagram of a pendulum](image)

**Newton's ODE**

$$mL\ddot{\theta} = -mg\sin\theta - mg\dot{\theta}^2$$

**damping constant**

$$\Rightarrow \ddot{\theta} = -\frac{g}{L}\sin\theta - c\dot{\theta}$$

1. Re-write this 2nd order ODE as a 1st order system:

$$\begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ -\frac{g}{L}\sin\theta - c\dot{\theta} \end{bmatrix} = f(x)$$

2. The critical points satisfy $\dot{\theta} = 0$

$$-\frac{g}{L}\sin\theta = 0 \Rightarrow \sin\theta = 0$$

And so are $[0], [\pm \pi], [\pm 2\pi], [\pm 3\pi], \ldots$, etc.

"down" & "up"
The Jacobian matrix is

\[ D^2 \mathcal{F}(\theta) = \begin{bmatrix} \frac{2}{c} (\theta') & \frac{2}{c} (\theta') \\ \frac{2}{c} (-\frac{c}{2} \sin(-\theta') \theta') & \frac{2}{c} (-\frac{c}{2} \sin(-\theta') \theta') \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{2} \cos \theta & -c \end{bmatrix} \]

Linearization at critical point \([0] \) ("down").

\[ J = D^2 \mathcal{F}(\theta) = \begin{bmatrix} 0 & 1 \\ -\frac{c}{2} & -c \end{bmatrix} \]

Eigenvalues: \[ \lambda = \text{det} \begin{bmatrix} -\lambda & 1 \\ -\frac{c}{2} & -c \end{bmatrix} = \lambda^2 + c \lambda + \frac{c^2}{4} = 0 \]

\[ \lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - \frac{c^2}{4}} \]

Case 1 (Overdamped): \( c^2 > \frac{c^2}{4} \) \( \Rightarrow \) 2 negative eigenvalues.

\[ \Rightarrow \text{sink \ (stable node \ which \ is \ \underline{asymptotically} \ stable} \]

Case 2 (Underdamped): \( c^2 < \frac{c^2}{4} \) \( \Rightarrow \) \( \lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - \frac{c^2}{4}} \)

Complex eigenvalues with negative real part.

\[ \Rightarrow \text{spiral sink, \ \underline{asymptotically} \ stable} \]

Note: \( J \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{c}{2} \end{bmatrix} \) \( \downarrow \), so direction of spiral is
at critical point \([\pi]\) ("up")

\[ J = \mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & -c \end{bmatrix} \]

whose eigenvalues are given by

\[ 0 = \det \begin{bmatrix} -\lambda & 1 \\ \frac{g}{L} & -a - c \end{bmatrix} = \lambda^2 + c\lambda - \frac{g}{L} \]

\[ \Rightarrow \lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + \frac{g}{L}} \]

2 eigenvalues of opposite sign

\[ \Rightarrow \text{ a saddle point, which is unstable} \]

(of course a pendulum pointing straight up is unstable!)

"eigenvectors:"

\[ \lambda = -\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{g}{L}} \]

\[ \begin{bmatrix} c_2 - \sqrt{\frac{c^2}{4} + \frac{g}{L}} & 1 \\ \frac{g}{L} & -\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{g}{L}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \lambda = -\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{g}{L}} \]

\[ \begin{bmatrix} c_2 + \sqrt{\frac{c^2}{4} + \frac{g}{L}} & 1 \\ \frac{g}{L} & -\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{g}{L}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

so the trajectories near \([\pi, 0]\) look like

\[ \text{Diagram} \]
"finally, we attempt to piece together the full phase portrait?"

Remark: notice there are a few very special trajectories (the incoming ones into the saddle points (the "ups") which have just the right initial velocity from a given starting position to exactly approach the "upward" position as $t \to \infty$. But this is very rare!

A "generic" choice of initial position and velocity will result in the pendulum settling down (perhaps after a few rotations) toward the "down" position.