Full Name: SOLUTIONS Signature: ____________________

Student Number: ____________________

Math 120 Midterm Test 3 Nov. 16, 2007 50 min.

1. To get full marks on the following short-answer questions, a correct answer must be written in the box provided.

(a) How fast is the side length of a cube increasing if its volume is 8 m^3, and the volume is increasing at a rate 1 m^3/s?

$\boxed{\frac{1}{12}\ m/s}$

$$l = V^{1/3}$$

$$\dot{l} = \frac{1}{3}V^{-2/3}\dot{V} = \frac{1}{3}8^{-2/3}\cdot 1 = \frac{1}{3\cdot 4} = \frac{1}{12}$$

(b) Find the slope of the curve $\sin^2(y)+2\cos^4(x) = 1$ at the point $(\pi/4, \pi/4)$.

$\boxed{2}$

$$2\sin(y)\cos(y)\frac{dy}{dx} + 8\cos^3(x)(-\sin(x)) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{8\cos^3(x)\sin(x)}{2\sin(y)\cos(y)} = 4\cdot\frac{(1/\sqrt{2})^3(1/\sqrt{2})}{(1/\sqrt{2})(1/\sqrt{2})} = \frac{4}{2} = 2$$

(c) Find the absolute minimum value of $2x^3+3x^2+1$ for $-2 \le x \le 2$.

$\boxed{-3}$

$$f(x) = 2x^3 + 3x^2 + 1$$

$$f'(x) = 6x^2 + 6x = 6x(x+1) = 0 \Leftrightarrow x = 0 \text{ or } x = -1$$

$$f(0) = 1$$

$$f(-1) = -2+3+1 = 2$$

$$f(-2) = -16+12+1 = -3 \Leftarrow$$

$$f(2) = 16+12+1 = 29$$

(d) Use a suitable linear approximation to find an estimate for $\sqrt{5}$ (given as a fraction).

$\boxed{9/4}$

$f(x) = x^{1/2}$ $\quad f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(4) = \frac{1}{2}\cdot\frac{1}{2} = 1/4$

$$\sqrt{5} = \sqrt{4+1} = f(4+1)$$
$$\approx f(4) + f'(4)\cdot 1$$
$$= 2 + 1/4 = \frac{9}{4}$$

(e) List (the x-coordinates of) all inflection points of the graph $y = \frac{1}{3}x^6 + \frac{1}{2}x^5 - \frac{5}{3}x^4 + 7x - 8$.

$\boxed{x = -2, \ x = 1}$

$$\frac{dy}{dx} = 2x^5 + \frac{5}{2}x^4 - \frac{20}{3}x^3 + 7$$
$$\frac{d^2y}{dx^2} = 10x^4 + 10x^3 - 20x^2 = 10x^2(x^2 + x - 2)$$
$$= 10x^2(x+2)(x-1)$$

$x=1$ inflection

$x=-2$ inflection

$x=0$ not inflection

(f) Compute $\frac{d}{dx}\tan^{-1}(\sin^{-1}(x))$.

$\boxed{(1 + [\sin^{-1}(x)]^2)^{-1}(1-x^2)^{-1/2}}$

$$= \frac{1}{1 + [\sin^{-1}(x)]^2} \cdot \frac{1}{\sqrt{1-x^2}}$$

2. Determine where the function $f(x) = \frac{1}{x^3 - x}$ is increasing/decreasing and concave up/down, find any asymptotes, and accurately sketch its graph.

$$f(x) = \frac{1}{x(x^2-1)}$$

- odd function
- vertical asymptotes at $x = 0, \pm 1$
- horizontal asymptote $y = 0$ as $x \to \pm\infty$
- $f'(x) = -\frac{1}{(x^3-x)^2}(3x^2-1) = \frac{1-3x^2}{(x^3-x)^2}$ $\Rightarrow$
  - decreasing on $(-\infty, -\frac{1}{\sqrt{3}}] \cup [\frac{1}{\sqrt{3}}, \infty)$ (excluding $x = \pm 1$)
  - increasing on $[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$ (excluding $x = 0$)
- $f''(x) = (x^3-x)^{-4}\left[(x^3-x)^2(-6x) - (1-3x^2)2(x^3-x)(3x^2-1)\right]$

$$= (x^3-x)^{-3}\left[6x(x-x^3) + 2(3x^2-1)^2\right]$$

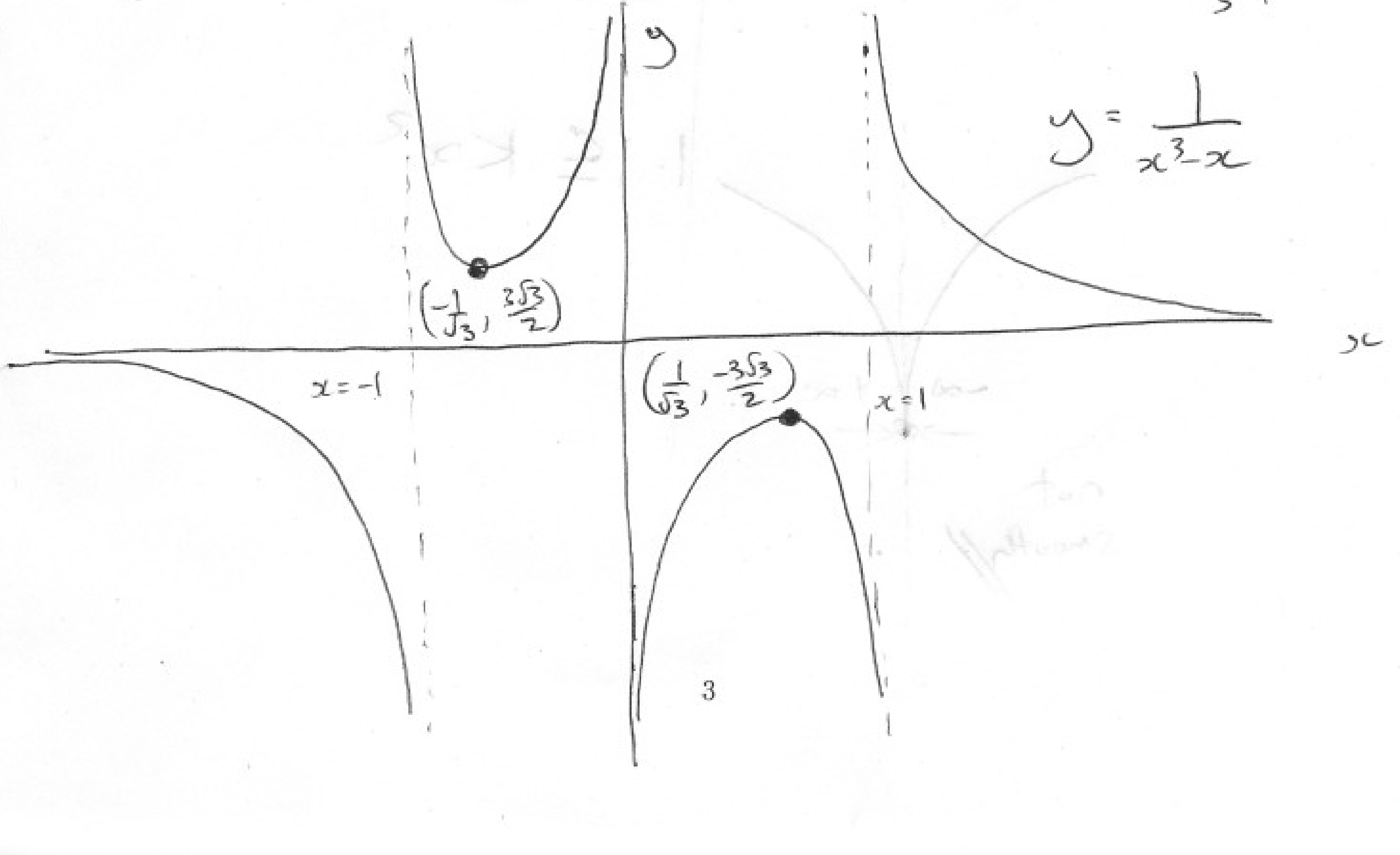
$$= 2(x^3-x)^{-3}\left[3x^2 - 3x^4 + 9x^4 - 6x^2 + 1\right]$$

$$= 2(x^3-x)^{-3}\left[6x^4 - 3x^2 + 1\right] = 2(x^3-x)^{-3}\Big[\overbrace{6(x^2-\tfrac{1}{4})^2 + \tfrac{5}{8}}^{>0}\Big]$$

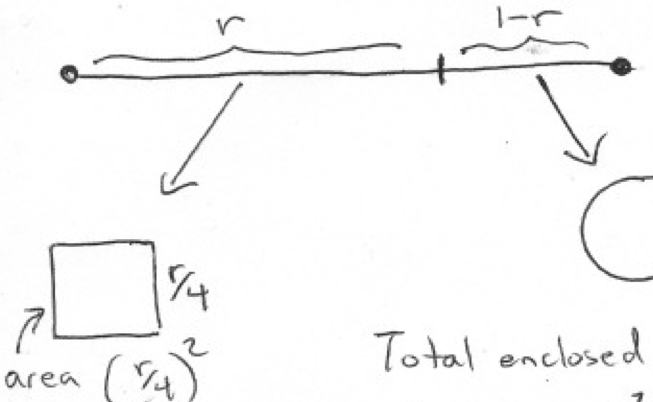
$\Rightarrow$ concavity determined by the sign of $x^3 - x = x(x^2-1)$

$\Rightarrow$ $\begin{cases} \text{concave up on } (-1, 0) \cup (1, \infty) \\ \text{concave down on } (-\infty, -1) \cup (0, 1) \end{cases}$

$$f\left(\tfrac{1}{\sqrt{3}}\right) = \frac{\sqrt{3}}{\frac{1}{3} - 1} = \frac{-3\sqrt{3}}{2}$$



3. A piece of wire is cut into 2 pieces. One piece is bent into a square, the other into a circle. What fraction of the wire should go to the square in order to (a) maximize the total (square plus circle) enclosed area? (b) minimize the total enclosed area?



(Spse. lenght of wire = 1)

perimeter = $1-r$ $\qquad 0 \le r \le 1$

radius = $\frac{1-r}{2\pi}$

$\leftarrow$ area = $\pi\left(\frac{1-r}{2\pi}\right)^2 = \frac{1}{4\pi}(1-r)^2$

Total enclosed area is

$$A(r) = r^2/16 + \frac{(1-r)^2}{4\pi} = \left(\frac{1}{16}+\frac{1}{4\pi}\right)r^2 - \frac{1}{2\pi}r + \frac{1}{4\pi}$$

Note $A(0) = \frac{1}{4\pi}$ (all circle) and $A(1) = \frac{1}{16}$ (all square),

so $A(0) > A(1)$.

Note $A'(r) = \left(\frac{1}{8}+\frac{1}{2\pi}\right)r - \frac{1}{2\pi}$ and $A''(r) = \frac{1}{8}+\frac{1}{2\pi} > 0$

$\Rightarrow$ CP at $r = \frac{1}{2\pi\left(\frac{1}{8}+\frac{1}{2\pi}\right)} = \frac{1}{\pi/4+1}$ $\qquad \Rightarrow$ any CP is a local min

$\Rightarrow$ max. is at endpoint $\qquad \Rightarrow$ $\boxed{r=0 \Leftrightarrow \text{all circle for maximum area}}$

The minimum must be at the critical point otherwise there would be a local max. in between the CP and $r=1$

$\Rightarrow$ $\boxed{r = \frac{1}{1+\pi/4} \text{ of the wire goes to the square for minimum area}}$

$$\left(\text{compare } A(1) - A\left(\frac{1}{1+\pi/4}\right) = \frac{1}{16} - \frac{1}{16(1+\pi/4)^2} - \frac{1}{4\pi}\left(\frac{\pi/4}{1+\pi/4}\right)^2\right.$$

$$\left. = 16(1+\pi/4)^{-2}\left\{(1+\pi/4)^2 - 1 - \frac{4}{\pi}\underbrace{\left(\frac{\pi}{4}\right)^2}_{\pi/4}\right\} = 16(1+\pi/4)^{-2}\left\{\frac{\pi}{4}+\left(\frac{\pi}{4}\right)^2\right\} > 0 \right)$$