Full Name: SOLUTIONS Signature: ____________

Student Number: ____________

Math 120 Midterm Test 3 Nov. 17, 2006 50 min.

1. For each of these short-answer questions, write your final answer in the box provided. *Only your final answer will be graded*, preliminary work will not be graded.

(a) Find dy/dx if $x^3 + y^3 = 6xy$.

$$\boxed{\frac{2y-x^2}{y^2-2x}}$$

$$3x^2 + 3y^2y' = 6(y+xy')$$

$$\Rightarrow (3y^2-6x)y' = 6y-3x^2 \Rightarrow y' = \frac{6y-3x^2}{3y^2-6x} = \frac{2y-x^2}{y^2-2x}$$

(b) Find the rate of change of the radius of a circle when its area is 100 cm^2, if the area is increasing at a rate 2 cm^2/min.

$$\boxed{\frac{1}{10\sqrt{\pi}}\ cm/_{min}}$$

- $A = \pi r^2$
- $A = 100\ cm^2 \Leftrightarrow r = \sqrt{\frac{A}{\pi}} = \frac{10\,cm}{\sqrt{\pi}}$

$$\frac{dA}{dt} = \pi \cdot 2r\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{2\pi r}\frac{dA}{dt} = \frac{1}{2\pi \cdot \frac{10cm}{\sqrt{\pi}}} \cdot 2\ \frac{cm^2}{min}$$

$$// \quad \frac{1}{10\sqrt{\pi}}\ cm/_{min}$$

(c) Find the approximation for $\sqrt{8}$ coming from the linearization of $f(x) = \sqrt{x}$ at $x = 9$.

$$\boxed{17/6}$$

$$f(9) = \sqrt{9} = 3$$

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(9) = \frac{1}{2}\cdot 9^{-1/2} = 1/6$$

$$\Rightarrow f(8) = f(9+-1) \approx f(9) + f'(9)(-1) = 3 - \tfrac{1}{6} = 17/6$$

(d) For which x is the graph $y = \frac{1}{12}x^4 - \frac{1}{6}x^3 - x^2 + 5x - 7$ concave up?

$$\boxed{(-\infty,-1] \cup [2,\infty)}$$

$$y' = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 5$$

$$y'' = x^2 - x - 2 = (x-2)(x+1)$$

$\Rightarrow y'' > 0$ for $x < -1$ and $x > 2$

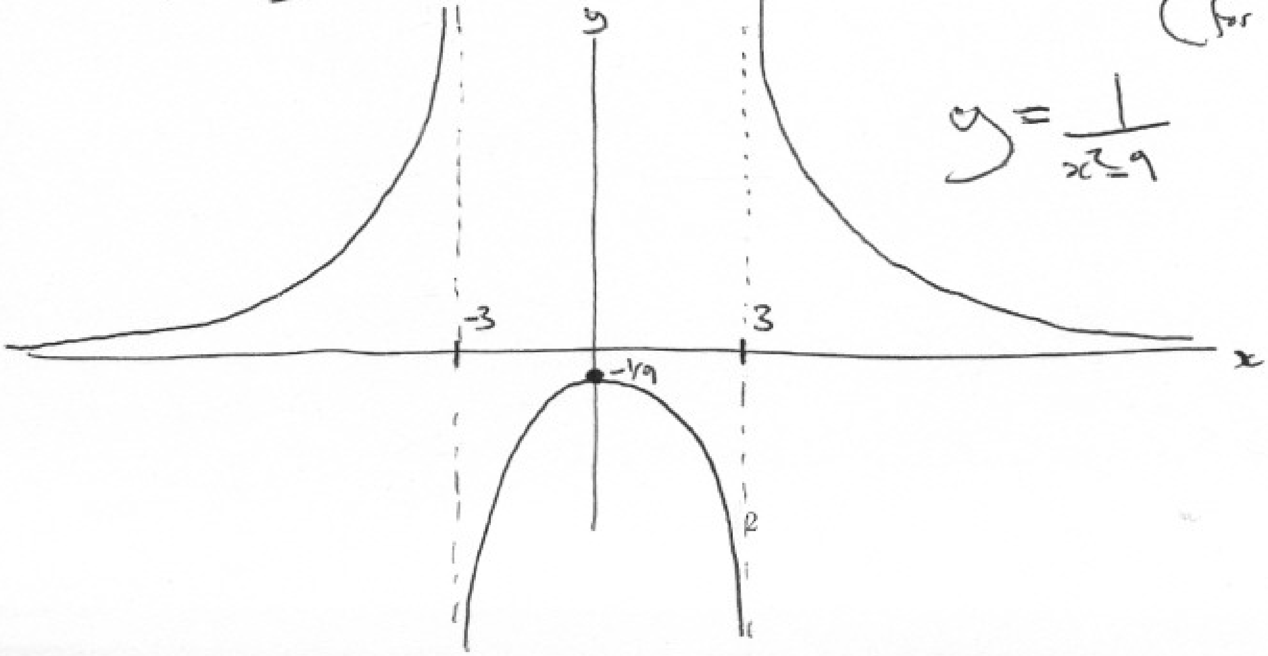
$\Rightarrow$ concave up (y' increasing) on $(-\infty,-1] \cup [2,\infty)$

2. Determine where the function

$$f(x) = \frac{1}{x^2 - 9}$$

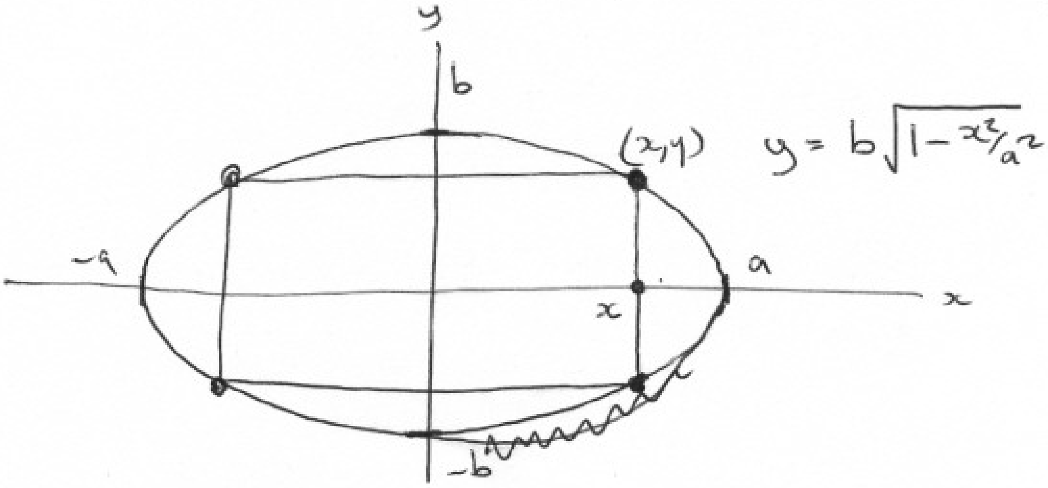
is increasing, decreasing, and concave up/down, find any asymptotes and intercepts, and make an accurate sketch of its graph.

- denominator $= 0$ (and numerator $\neq 0$) at $x = \pm 3$
  $\Rightarrow$ vertical asymptotes there
- no x intercepts ($f \neq 0$); y intercept : $f(0) = \frac{1}{-9} = -1/9$
- $\lim_{x \to \pm\infty} f(x) = 0 \Rightarrow y = 0$ is horizontal asymptote (both directions)
- $f'(x) = -\frac{1}{(x^2-9)^2} \cdot 2x \Rightarrow \begin{cases} f' > 0 \text{ (increasing) for } x < 0 \\ f' < 0 \text{ (decreasing) for } x > 0 \end{cases}$
- $f''(x) = \frac{-2}{(x^2-9)^2} - 2x \cdot (-2)(x^2-9)^{-3} \cdot 2x$
  $= 2(x^2-9)^{-3}\left[-(x^2-9) + 4x^2\right] = \frac{3x^2+9}{(x^2-9)^3} \Rightarrow \begin{cases} f'' > 0 \text{ (concave up) for } x < -3,\ x > 3 \\ f'' < 0 \text{ (concave down) for } -3 < x < 3 \end{cases}$
- also f is <u>even</u>



3. Find the area of the largest rectangle that can be inscribed (with sides parallel to the axes) in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



- let $x > 0$ be the intersection of the rectangle with x-axis (so $2x$ is the width)
- then the height is $2 \cdot y = 2b\sqrt{1 - x^2/a^2}$

$\Rightarrow$ the area is $$A(x) = 2x \cdot 2b\sqrt{1 - x^2/a^2} = 4b\, x\left(1 - x^2/a^2\right)^{1/2}$$

$$\text{for } 0 \le x \le a$$

- endpoints: $A(0) = A(a) = 0$
- critical points: $$0 = A'(x) = 4b\left[\left(1 - x^2/a^2\right)^{1/2} + x \cdot \tfrac{1}{2}\left(1 - x^2/a^2\right)^{-1/2} \cdot \frac{-2x}{a^2}\right]$$
$$= 4b\left(1 - x^2/a^2\right)^{-1/2}\left[1 - x^2/a^2 - x^2/a^2\right] = 4b\,\frac{1 - 2x^2/a^2}{\sqrt{1 - x^2/a^2}}$$

$\Rightarrow x^2 = a^2/2 \Rightarrow x = a/\sqrt{2}$

and $A(a/\sqrt{2}) = 4b \cdot \frac{a}{\sqrt{2}} \cdot \sqrt{1 - 1/2} = \boxed{2ab}$

- since $A(x)$ must have an abs. max. on $[0, a]$ (cont. function on a closed, bounded interval), this must be it.

4. Find the critical points and singular points (points where the derivative is not defined) of the function $f(x) = x^{2/3} - (2x-8)^{2/3}$. Find its absolute maximum and minimum values, if they exist.

• f is not differentiable at $\boxed{x=0}$ and $2x-8=0 \Leftrightarrow \boxed{x=4}$

(these are the singular points)

$$f(0) = -(-8)^{2/3} = -8^{2/3} = -4$$

$$f(4) = 4^{2/3}$$

• critical points?: $0 = f'(x) = \frac{2}{3}x^{-1/3} - \frac{2}{3}(2x-8)^{-1/3} \cdot 2$

$$\Leftrightarrow x^{-1/3} = 2(2x-8)^{-1/3}$$

$$\Leftrightarrow (2x-8)^{1/3} = 2x^{1/3} = (8x)^{1/3}$$

$$\Leftrightarrow 2x-8 = 8x \Leftrightarrow 6x = -8 \Leftrightarrow \boxed{x = -\tfrac{4}{3}} \quad \text{(critical point)}$$

$$f\left(-\tfrac{4}{3}\right) = \left(-\tfrac{4}{3}\right)^{2/3} - \left(-\tfrac{8}{3}-8\right)^{2/3} = \left(\tfrac{4}{3}\right)^{2/3} - \left(\tfrac{32}{3}\right)^{2/3} < 0$$

• note as $x \to \infty$ $f(x) \approx x^{2/3} - (2x)^{2/3} = (1-2^{2/3})x^{2/3} \to -\infty$

and as $x \to -\infty$, $f(x) \approx |x|^{2/3} - (2|x|)^{2/3} = (1-2^{2/3})|x|^{2/3} \to -\infty$

$\Rightarrow$ $\boxed{\text{no absolute } \cancel{\text{maximum}} \text{ min.}}$, but there must be an absolute max. which (comparing values at singular and critical points) is $\boxed{f(4) = 4^{2/3} \text{ abs. max.}}$