Full Name: ____________________ Signature: ____________________

Student Number: ____________________

# Math 120 Midterm Test 2 Oct. 26, 2007 50 min.

$\frac{\ }{18}$

1. Each of these short-answer questions is worth 3 marks. If a correct answer is written in the box provided, you get 3, otherwise you can earn at most 1.

(a) Express ln(3/4) in terms of ln(3) and ln(2).

$\boxed{\ln(3) - 2\ln(2)}$

$$\ln(3/4) = \ln(3) - \ln(2^2)$$
$$= \ln(3) - 2\ln(2)$$

(b) Compute $\frac{d}{dx}e^{-\cos(x^2)}$.

$\boxed{e^{-\cos(x^2)}\sin(x^2)2x}$

$$= e^{-\cos(x^2)}(-(-\sin(x^2))\cdot 2x = e^{-\cos(x^2)}\sin(x^2)2x$$

(c) Let $f$ be a one-to-one differentiable function with $f(1) = 2$, $f(2) = 1$, $f'(1) = 2$, $f'(2) = 3$. Let $g(x) = f^{-1}(x^2)$. Find $g'(1)$.

$\boxed{2/3}$

$$g'(1) = 2x \cdot (f^{-1})'(x^2)\Big|_{x=1} = 2 \cdot \frac{1}{f'(f^{-1}(1))} = \frac{2}{f'(2)} = \frac{2}{3}$$

(d) Find an equation for the tangent line to the graph $y = \tan(x)$ at $x = 0$. $\boxed{y = x}$

$$\left.\frac{dy}{dx}\right|_{x=0} = \sec^2(x)\Big|_{x=0} = \frac{1}{\cos^2(0)} = 1$$

$\tan(0) = 0$

$\Rightarrow$ tangent line is $y = x$

(e) Compute $\frac{d}{dx}[\ln(x)]^{x^2}$. $\boxed{[\ln(x)]^{x^2} x\left(2\ln(\ln(x)) + 1/\ln(x)\right)}$

$$y := [\ln(x)]^{x^2}$$

$$\ln(y) = x^2 \ln(\ln(x))$$

$$\Rightarrow \frac{y'}{y} = 2x\ln(\ln(x)) + x^2 \cdot \frac{1}{\ln(x)\cdot x}$$

$$\Rightarrow y' = [\ln(x)]^{x^2} x\left(2\ln(\ln(x)) + \frac{1}{\ln(x)}\right)$$

(f) Find lim)$x \to 0 \left[\frac{2\sin(x)}{x\sin(2x)} - \frac{1}{x}\right]$. $\boxed{0}$

$$\lim_{x\to 0} \frac{1}{x}\left[\frac{2\sin(x)}{\sin(2x)} - 1\right] = \lim_{x\to 0} \frac{1}{x}\left[\frac{2\cancel{\sin(x)}}{2\cancel{\sin(x)}\cos(x)} - 1\right]$$

$$= \lim_{x\to 0} \frac{1-\cos(x)}{x\cos(x)} = \lim_{x\to 0} \frac{1 - [1 - 2\sin^2(x/2)]}{x\cos(x)}$$

$$= \lim_{x\to 0} \left[\frac{\sin(x/2)}{x/2}\right] \frac{\sin(x/2)}{2\cos(x)} = \underbrace{\left[\lim_{x\to 0} \frac{\sin(x/2)}{x/2}\right]}_{1}{}^{2} \cdot \frac{\overbrace{\lim_{x\to 0} \sin(x/2)}^{=0}}{\underbrace{2\lim_{x\to 0}\cos(x)}_{1}}$$

$$= 0$$

(5)

2. An object moving along the real line has initial (time 0) position 1 , initial velocity 1, and acceleration $a(t) = \sin(2t)$. Find its position as a function of time.

$$v(t) = -\frac{1}{2}\cos(2t) + C$$

$$v(0) = 1 \Rightarrow -\frac{1}{2} + C = 1 \Rightarrow C = 3/2$$

$$x(t) = -1/4 \sin(2t) + 3/2\, t + D$$

$$x(0) = 1 \Rightarrow D = 1$$

$$\Rightarrow \boxed{x(t) = -1/4 \sin(2t) + \frac{3}{2}t + 1}$$

(5)

3. Solve the initial value problem

$$\begin{cases} dy/dt = 3y - 2 \\ y(1) = 2 \end{cases}$$

to find $y(t)$.

Set $x(t) = 3y(t) - 2$, so $\frac{dx}{dt} = 3\frac{dy}{dt} = 3x$.

Hence $x(t) = Ce^{3t}$.

$$x(1) = 3y(1) - 2 = 3 \cdot 2 - 2 = 4 \Rightarrow 4 = Ce^{3}$$

$$\Rightarrow C = 4e^{-3}.$$

So $x(t) = 4e^{3(t-1)}$

and so $y(t) = \frac{1}{3}[x(t) + 2] = \boxed{4/3\, e^{3(t-1)} + 2/3}$

(6) 4. Let $f(x) = 2x\ln(x)$. Find the largest interval on which $f$ is a one-to-one function. Compute $[f^{-1}]'(2e)$.

$f'(x) = 2(\ln(x) + 1) = 0$ if $\ln(x) = -1$

$\iff x = e^{-1} = 1/e$

So $f'(x) \geq 0$ on $[\frac{1}{e}, \infty)$, $f'(x) \leq 0$ on $(0, \frac{1}{e}]$.

So the largest interval on which $f$ is one-to-one is $\boxed{[\frac{1}{e}, \infty)}$

$$(f^{-1})'(2e) = \frac{1}{f'(f^{-1}(2e))} = \frac{1}{f'(e)} = \frac{1}{2(\ln(e)+1)} = \frac{1}{2\cdot 2} = \boxed{\frac{1}{4}}$$

(6) 5. State the mean value theorem (MVT). Does the function $f(x) = x^{1/3}$ on the interval $[-1, 1]$ satisfy the hypotheses of MVT? Does it staisfy the conclusions of MVT?

- MVT: If $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then $\exists\ c \in (a,b)$ s.t. $\frac{f(b)-f(a)}{b-a} = f'(c)$.

- the MVT does <u>not</u> apply to $f(x) = x^{1/3}$ on $[-1,1]$ since $f$ is not differentiable at $x = 0 \in (-1,1)$.

- however, $\frac{f(1)-f(-1)}{1-(-1)} = \frac{1-(-1)}{1-(-1)} = 1$, and we can solve

$$f'(x) = \frac{1}{3}x^{-2/3} = 1 \Rightarrow x^{-2/3} = 3 \iff x = \pm 3^{-3/2} = \pm\frac{1}{3^{3/2}}.$$

Since both $\frac{1}{3^{3/2}}$ and $\frac{-1}{3^{3/2}}$ lie in $(-1,1)$, the <u>conclusion</u> of MVT is still true.