Full Name: SOLUTIONS Signature: ______________________

Student Number: ______________________

Math 120 Midterm Test 2 Oct. 27, 2006 50 min.

1. For each of these short-answer questions, write your final answer in the box provided. *Only your final answer will be graded*, preliminary work will not be graded.

   (a) Simplify $e^{a\ln(b)-\ln(c)}$.

   $\boxed{b^a/c}$

   $$= e^{\ln(b^a)-\ln(c)}$$
   $$= e^{\ln(b^a/c)} = b^a/c$$

   (b) Find $(f^{-1})'(e)$ if $f(x) = e^{x^2}$ for $x \geq 0$

   $\boxed{\frac{1}{2e}}$

   $$f(1) = e^1 = e \Rightarrow f^{-1}(e) = 1$$
   $$f'(x) = 2xe^{x^2}$$
   $$(f^{-1})'(e) = \frac{1}{f(f^{-1}(e))} = \frac{1}{f'(1)} = \frac{1}{2\cdot 1\cdot e^1} = \frac{1}{2e}$$

   (c) Find $\lim_{x\to 0} \frac{x\cos(x)}{\sin(2x)}$.

   $\boxed{1/2}$

   $$= \lim_{x\to 0} \frac{x\cancel{\cos(x)}}{2\sin(x)\cancel{\cos(x)}} = \frac{1}{2}\lim_{x\to 0}\frac{x}{\sin(x)} = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

(d) Find the n-th derivative, $f^{(n)}(x)$, of $f(x) = e^{-3x}$.

$\boxed{(-3)^n e^{-3x}}$

$f' = (-3)e^{-3x}$

$f'' = (-3)^2 e^{-3x}$

$\vdots$

$f^{(n)}(x) = (-3)^n e^{-3x}$

(e) Evaluate $\cos^{-1}(\cos(\pi + 1/7))$.

$\boxed{\pi - 1/7}$

$\cos(\pi + 1/7) = -\cos(\tfrac{1}{7}) = -\cos(-1/7)$

$= \cos(\pi - \tfrac{1}{7})$

and $\pi - \frac{1}{7} \in [0, \pi] \Rightarrow \cos^{-1}(\cos(\pi + \tfrac{1}{7})) = \pi - \tfrac{1}{7}$

(f) Compute $\frac{d}{dt} e^{\sin(\ln(t))}$.

$\boxed{e^{\sin(\ln(t))} \cos(\ln(t))/t}$

$= e^{\sin(\ln(t))} \cdot \cos(\ln(t)) \cdot \frac{1}{t}$

2. Solve the initial value problem

$$y'' = \cos(2x) + x, \qquad y(0) = 0, \;\; y'(0) = 1.$$

$$y'(x) = \frac{1}{2}\sin(2x) + \frac{1}{2}x^2 + C$$

$$1 = y'(0) = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0 + C = C \Rightarrow C = 1$$

$$\Rightarrow y'(x) = \frac{1}{2}\sin(2x) + \frac{1}{2}x^2 + 1$$

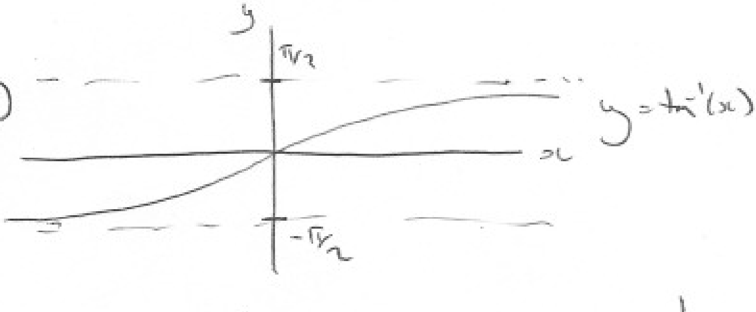
$$\Rightarrow y(x) = -\frac{1}{4}\cos(2x) + \frac{1}{6}x^3 + x + D$$

$$0 = y(0) = -\frac{1}{4} + 0 \Rightarrow D = 1/4$$

$$\Rightarrow \boxed{y(x) = -\frac{1}{4}\cos(2x) + \frac{1}{6}x^3 + x + 1/4}$$

3. (a) Sketch the graph $y = \tan^{-1}(x)$.
(b) Show that $\tan^{-1}(x) + 1 \geq \frac{1}{1+x^2}$ for $x \geq 0$.

(a)



(b) Set $f(x) = \tan^{-1}(x) + 1 - \frac{1}{1+x^2}$.

Then $f(0) = 0 + 1 - 1 = 0$.

And $f'(x) = \frac{1}{1+x^2} + \frac{2x}{(1+x^2)^2} > 0$ for $x \geq 0$

$\Rightarrow$ $f(x)$ is increasing for $x \geq 0$ $\Rightarrow$ $f(x) \geq f(0) = 0$.

4. Suppose an object falling under gravity experiences a force of air resistance proportional to its velocity (but in the opposite direction). Then Newton's equation, written in terms of the (downward) velocity $v(t)$, is

$$\frac{dv}{dt} = g - kv = k(\frac{g}{k} - v)$$

where g and k are positive constants. Suppose the object is dropped from rest at time $t = 0$.

(a) Find $v(t)$, the velocity as a function of time (in terms of g and k).

(b) Pretending the object never hits the ground, what is the "terminal velocity" $\lim_{t\to\infty} v(t)$?

a) set $y(t) = g/k - v(t)$, Then $y(0) = g/k - v(0) = g/k$

and $\frac{dy}{dt} = -\frac{dv}{dt} = -ky$. So $y(t) = g/k\, e^{-kt}$ and

$v(t) = g/k - g/k\, e^{-kt} = \boxed{g/k(1 - e^{-kt})}$.

b) $\lim_{t\to\infty} v(t) = \boxed{g/k}$

5. Let $f(x) = x^x$ restricted to the domain $[\frac{1}{e}, \infty)$.

(a) Show that f is one-to-one on this domain.

(b) Find $[f^{-1}]'(4)$.

a) $\ln(f) = x\ln(x)$

$\Rightarrow \frac{f'}{f} = (x\ln(x))' = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$

$\Rightarrow f'(x) = x^x(\ln(x)+1)$.

For $x > \frac{1}{e}$, $\ln(x) > \ln(\frac{1}{e}) = -\ln(e) = -1 \Rightarrow \ln(x) + 1 > 0$

and so $f'(x) > 0$ and f is increasing on $[\frac{1}{e}, \infty)$,

hence one-to-one.

(b) $f(2) = 2^2 = 4 \Rightarrow f^{-1}(4) = 2$. and $(f^{-1})'(4) = \frac{1}{f'(2)} = \frac{1}{2^2(\ln(2)+1)}$

$= \boxed{\frac{1}{4(\ln(2)+1)}}$