Full Name: Solutions (version 1) Signature: ____________

Student Number: ____________

Math 120 Midterm Test 1 Sep. 29, 2006 50 min.

1. For each of these short-answer questions, write your final answer in the box provided. *Only your final answer will be graded*, any preliminary work will not be looked at.

(a) Find $\lim_{x\to -3} \frac{|x|(x+3)}{x^2+4x+3}$

$\boxed{-3/2}$

$$= \lim_{x\to -3} \frac{|x|(x+3)}{(x+3)(x+1)}$$

$$= \lim_{x\to -3} \frac{|x|}{x+1} = \frac{|-3|}{-3+1} = \frac{3}{-2}$$

(b) Compute the derivative of $f(x) = \frac{1-x^2}{1+x^2}$

$\boxed{-4x/(1+x^2)^2}$

$$f'(x) = \frac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2}$$

$$= \frac{-4x}{(1+x^2)^2}$$

(c) Find an equation for the tangent line to $y = \sqrt{1+3x}$ at $x = 1$

$\boxed{y = \frac{3}{4}x + \frac{5}{4}}$

$$\left.\begin{aligned} f(x) &= (1+3x)^{1/2} \\ f(1) &= 4^{1/2} = 2 \\ f'(x) &= \tfrac{1}{2}(1+3x)^{-1/2} \cdot 3 \\ \Rightarrow f'(1) &= \tfrac{3}{2} \cdot 4^{-1/2} = 3/4 \end{aligned}\right\} \Rightarrow$$ tangent line is

$$y = 3/4(x-1) + 2$$

$$= 3/4 x + 5/4$$

(d) Find all numbers b such that the function

$$f(x) = \begin{cases} x(x+b) & x \le 0 \\ x & 0 < x \le 1 \\ (x-b+1)^2 & x > 1 \end{cases}$$

is continuous everywhere.

$\boxed{b = 1 \text{ or } 3}$

For continuity, we need

$f(0) = \lim_{x\to 0-} f = \lim_{x \to 0+} f \Leftrightarrow 0 = 0 = 0$ ✓

and $f(1) = \lim_{x \to 1-} f = \lim_{x\to 1+} f \Leftrightarrow 1 = 1 = (2-b)^2 \Rightarrow 2-b = \pm 1$

$\Rightarrow b = 1 \text{ or } 3$

(e) Find $h'(1)$, if $h(x) = \sqrt{f(x^2 g(x))}$, where f and g are differentiable functions satisfying $f(3) = 4$, $f'(3) = 1$, $g(1) = 3$, and $g'(1) = 1$

$\boxed{7/4}$

$$h'(1) = \frac{1}{2}\Big[\underbrace{f(1\cdot \underbrace{g(1)}_{3})}_{4}\Big]^{-1/2} \cdot \underbrace{f'(1 \cdot \underbrace{g(1)}_{3})}_{1} \cdot \big(2 \cdot 1\, \underbrace{g(1)}_{3} + 1 \cdot \underbrace{g'(1)}_{1}\big)$$

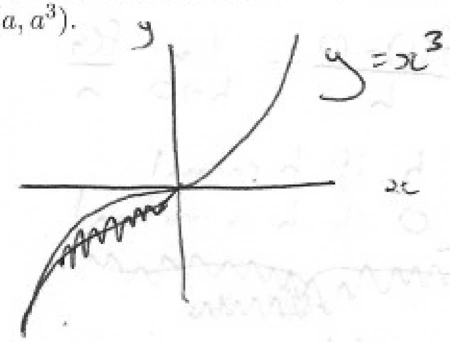
$$= \frac{1}{2} 4^{-1/2} \cdot 1 \cdot (6+1) = \frac{1}{4} \cdot 7$$

(f) Find $\lim_{x\to-\infty} \frac{4x+2}{\sqrt{x^2-x-1}}$

$\boxed{-4}$

$$= \lim_{x \to -\infty} \frac{4x+2/|x|}{\sqrt{x^2-x-1}/|x|} = \lim_{x\to-\infty} \frac{-4 + 2/|x|}{\sqrt{1 - \frac{1}{x} - \frac{1}{x^2}}} = \frac{-4}{\sqrt{1}} = -4$$

2. Sketch (roughly) the graph $y = x^3$. Find an equation for the line *normal* to this graph at a point (a, a^3).



$$\left.\frac{dy}{dx}\right|_{x=a} = \left.3x^2\right|_{x=a} = 3a^2$$

so the slope of the <u>normal</u> line is

$$-\frac{1}{3a^2}$$

$\Rightarrow$ an equation for the normal line is

$$y = -\frac{1}{3a^2}(x-a) + a^3$$

or $$\boxed{y = \frac{-1}{3a^2}x + \frac{1}{3a} + a^3}$$

(if $a = 0$, the normal line is vertical, i.e. $\boxed{x = 0}$)

3. Use the definition of limit to prove that

$$\lim_{x \to 1} (x-1)^3 = 0.$$

Let $\varepsilon > 0$ be given.

Suppose $0 < |x-1| < \delta$.

Then $|(x-1)^3 - 0| = |x-1|^3 < \delta^3$.

So if we choose $\delta = \varepsilon^{1/3}$, we have

$$|(x-1)^3 - 0| < \varepsilon.$$

Hence $\lim_{x \to 1} (x-1)^3 = 0$.

4. Is the function

$$f(x) := \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

differentiable at $x = 0$? (Justify your answer.)

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

$$= \lim_{h \to 0} \begin{cases} h & \text{if } h \text{ rational} \\ 0 & \text{if } h \text{ is irrational} \end{cases}$$

Note that $-|h| \le \frac{f(h)}{h} \le |h|$ and $\lim_{h \to 0} -|h| = \lim_{h \to 0} |h| = 0$.

So by the "squeeze theorem", $\lim_{h \to 0} \frac{f(h)}{h} = 0$.

So f is differentiable at $x = 0$, with $f'(0) = 0$.

5. Let $f_1(x)$, $f_2(x)$, $\ldots$, $f_n(x)$ be n differentiable functions satisfying $f_j(1) = 2$ and $f_j'(1) = 3$ for all $j = 1, 2, \ldots, n$. Let $g(x)$ be the product of these n functions:

$$g(x) := f_1(x) f_2(x) \cdots f_n(x).$$

Find $g'(1)$.

By the product rule,

$$g'(1) = f_1'(1) f_2(1) \cdots f_n(1) + f_1(1) f_2'(1) f_3(1) \cdots f_n(1)$$

$$+ \cdots + f_1(1) f_2(1) \cdots f_{n-1}(1) f_n'(1) \quad (n \text{ terms}).$$

Each of these n terms $= \left\{ \begin{array}{l} 1 \text{ factor of } f_j'(1) = 3 \\ \text{and } (n-1) \text{ factors of } f_j(1) = 2 \end{array} \right\} = 3 \cdot 2^{n-1}$

$$\Rightarrow \boxed{g'(1) = 3n2^{n-1}}$$