Full Name: SOLUTIONS Signature: ______________________
Student Number: ______________________

Math 120 Midterm Test 1 Sep. 28, 2007 50 min.

1. Each of these short-answer questions is worth 3 marks. If a correct answer is written in the box provided, you get 3, otherwise you can earn at most 1.

(a) Find $\lim_{x\to 4} \frac{x^{3/2}-4x^{1/2}}{x^2-3x-4}$

$\boxed{2/5}$

$$= \lim_{x\to 4} \frac{x^{1/2}(x-4)}{(x-4)(x+1)}$$

$$= \lim_{x\to 4} \frac{x^{1/2}}{x+1} = \frac{2}{5}$$

(b) Compute the derivative of $f(x) = x\sqrt{1+x^4}$

$\boxed{\frac{1+3x^4}{\sqrt{1+x^4}}}$

$$f(x) = x(1+x^4)^{1/2}$$

$$f'(x) = (1+x^4)^{1/2} + x\tfrac{1}{2}(1+x^4)^{-1/2}4x^3$$

$$= (1+x^4)^{-1/2}\left[1+x^4+2x^4\right]$$

(c) Find an equation for the tangent line to $y = 1/x^2$ at $x = 2$.

$\boxed{y = -\frac{1}{4}x + 3/4}$

$$y\big|_{x=2} = \frac{1}{4}$$

$$\frac{dy}{dx}\bigg|_{x=2} = \frac{-2}{x^3}\bigg|_{x=2} = -\frac{1}{4}$$

$\Rightarrow$ tangent line is $y - 1/4 = (-1/4)(x-2)$

$$\Leftrightarrow y = -1/4\,x + 3/4$$

(d) For which integer values of k and b is the function

$$f(x) = \begin{cases} \frac{x^k}{|x|} & x \neq 0 \\ b & x = 0 \end{cases}$$

continuous everywhere?

$\boxed{b=0 \text{ and } k>1}$

- f cont. for $x \neq 0$ $\forall$ b and k
- f cont. at $0 \iff b = f(0) = \lim_{x\to 0} f(x) = \lim_{x \to 0} \frac{x^k}{|x|} = \begin{cases} 0 & \text{if } k>1 \\ \text{does not exist} & \text{if } k \leq 1 \end{cases}$

$\Rightarrow$ f cont. everywhere iff $b=0$ and $k>1$

(e) Find $h'(1)$, if $h(x) = f(g(f(x)))$, where f and g are differentiable functions satisfying

$f(1) = 2$, $f'(1) = 3$, $g(2) = 1$, and $g'(2) = 2$

$\boxed{18}$

$h'(x) = f'(g(f(x))g'(f(x))f'(x)$

$\Rightarrow h'(1) = f'(\overbrace{g(2)}^{1})\,g'(2)\,f'(1) = 3\cdot 2\cdot 3 = 18$

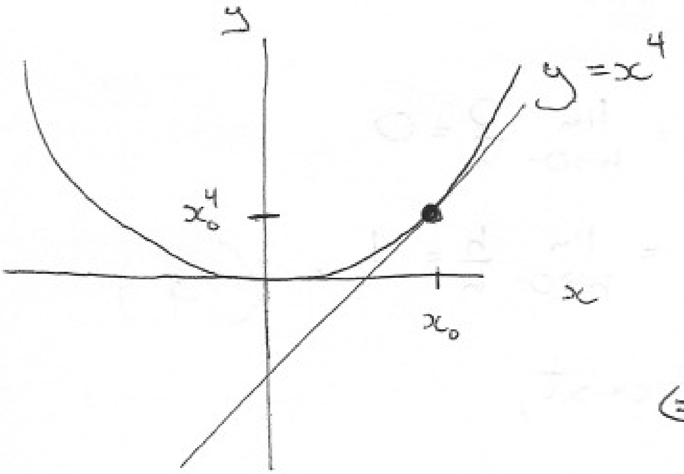
(f) Find $\lim_{x\to 1-} \frac{|x-1|}{x^2-1}$

$\boxed{-1/2}$

$= \lim_{x \to 1-} \frac{-(x-1)}{(x-1)(x+1)}$

$= \lim_{x\to 1-} \frac{-1}{x+1} = -1/2$

2. Sketch the graph $y = x^4$. Find equations for all tangent lines to this graph which pass through the point $(0, -48)$.



• $\frac{dy}{dx} = 4x^3$, so the tangent line at (x_0, x_0^4) is

$$y - x_0^4 = 4x_0^3(x - x_0)$$

$$\iff y = 4x_0^3 x - 3x_0^4$$

which intersects the y-axis at $y = -3x_0^4$. So we need $-3x_0^4 = -48$

$$\iff x_0^4 = 16$$

$$\iff x_0 = \pm 2.$$

So the lines are

$$\boxed{\begin{array}{l} y = 32x - 48 \\ y = -32x - 48 \end{array}}$$

3. Let

$$f(x) = \frac{x^3 + 3x + 6}{\sqrt{1 + x^2}}.$$

Compute the derivative $f'(x)$. Also, show that $f'(x) \geq 0$ for all x.

$$f'(x) = \frac{(1+x^2)^{1/2}(3x^2+3) - (x^3+3x+6)\frac{1}{2}(1+x^2)^{-1/2}(2x)}{1+x^2}$$

$$= (1+x^2)^{-3/2}\left[(1+x^2)(3x^2+3) - (x^4+3x^2+6x)\right]$$

$$= (1+x^2)^{-3/2}\left[2x^4 + 3x^2 - 6x + 3\right]$$

$$= \underbrace{(1+x^2)^{-3/2}}_{>0}\Big[\underbrace{2x^4}_{\geq 0} + \underbrace{3(x-1)^2}_{\geq 0}\Big] \geq 0.$$

4. Is the function

$$f(x) := \begin{cases} 0 & x \le 0 \\ x & x > 0 \end{cases}$$

differentiable at $x = 0$? (Justify your answer.)

$$\lim_{h \to 0-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0-} \frac{f(h)}{h} = \lim_{h \to 0-} \frac{0}{h} = 0$$

$$\lim_{h \to 0+} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0+} \frac{f(h)}{h} = \lim_{h \to 0-} \frac{h}{h} = 1 .$$

(4)

So $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$ does not exist,

hence f is <u>not</u> differentiable at 0.

5. Prove that the limit of a sum is the sum of the limits. That is, prove (using the definition of limit) that if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a}[f(x) + g(x)] = L + M$.

Let $\varepsilon > 0$ be given.

Since $\lim_{x \to a} f(x) = L$, $\exists\ \delta_1 > 0$ s.t. if $0 < |x-a| < \delta_1$, then $|f(x) - L| < \varepsilon/2$.

Since $\lim_{x \to a} g(x) = M$, $\exists\ \delta_2 > 0$ s.t. if $0 < |x-a| < \delta_2$, then $|g(x) - M| < \varepsilon/2$.

Set $\delta := \min(\delta_1, \delta_2)$. Then if $0 < |x-a| < \delta$,

$$|f(x)+g(x) - (L+M)| = |f(x) - L + g(x) - M| \le |f(x)-L| + |g(x)-M| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $\lim_{x \to a} [f(x)+g(x)] = L+M$.

(5)