Assignment 9 Solutions

1) $f(x) = x^p, \quad f'(x) = px^{p-1}$

$\Rightarrow$ Newton's method is $\quad x_{n+1} = x_n - \dfrac{x_n^p}{p x_n^{p-1}} = \left(1-\frac{1}{p}\right)x_n$

$x_1 = \left(1-\frac{1}{p}\right)x_0, \quad x_2 = \left(1-\frac{1}{p}\right)^2 x_0, \ \ldots \quad \boxed{x_n = \left(1-\frac{1}{p}\right)^n x_0}$

This will converge to 0 $\Leftrightarrow \left|1-\frac{1}{p}\right| < 1 \Leftrightarrow -1 < 1-\frac{1}{p} < 1$

$\Leftrightarrow -2 < -\frac{1}{p} < 0$

$\Leftrightarrow \frac{1}{2} < p < \infty$

$\Leftrightarrow \boxed{p > \frac{1}{2}}$

• $p=2, x_0=1 \Rightarrow x_n = \dfrac{1}{2^n} < 10^{-4}$

$\Leftrightarrow 2^n > 10^4 \Leftrightarrow n\ln(2) > 4\ln(10)$

$\Leftrightarrow n > \dfrac{4\ln(10)}{\ln(2)} \approx 13.3 \Rightarrow$ need $\boxed{\text{14 iterations}}$

2) $f(x) = x^2-1$, $f'(x) = 2x$ $\Rightarrow$ Newton is $x_{n+1} = x_n - \dfrac{x_n^2-1}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$

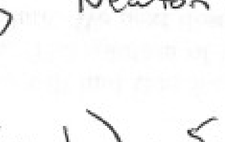
a) $x_0 = 2, \quad x_1 = \frac{1}{2}\left(2+\frac{1}{2}\right) = 5/4 = 1 + 1/4$

$x_2 = \frac{1}{2}\left(\frac{5}{4}+\frac{4}{5}\right) = \frac{41}{40} = 1 + 1/40$

$x_3 = \frac{1}{2}\left(\frac{41}{40}+\frac{40}{41}\right) \approx 1.000305$

$x_4 \approx 1.000000046$ $\Rightarrow \boxed{\text{4 iterations}}$

b) In question 1, the root is at a critical point $(f'=0)$ wheras here, the root is at a point of non-zero slope



c) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$ $\begin{cases} \text{converges to } 1 & \text{if } x_0 > 0 \\ \text{converges to } -1 & \text{if } x_0 < 0 \\ \text{is undefined} & \text{if } x_0 = 0 \end{cases}$

For example, suppose $x_0 > 0$. Then $x_n > 0 \;\forall n$.

<u>Claim:</u> $x_n \geq 1 \;\forall\, n \geq 1$

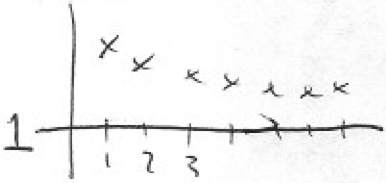
<u>Pf</u>: ~~[illegible]~~

Arithmetic-geometric mean inequality: $\underbrace{\frac{1}{2}\left(x_n + \frac{1}{x_n}\right)}_{= x_{n+1}} \geq \sqrt{x_n} \cdot \sqrt{\frac{1}{x_n}} = 1$.

<u>Claim:</u> $x_{n+1} \leq x_n \;\forall\; n \geq 1$.

<u>Pf</u>: $x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) = \frac{1}{2}\left(x_n - \frac{1}{x_n}\right) \geq 0$ since $x_n \geq 1$.

So the sequence $\{x_n\}_{n=1}^{\infty}$ is
- non-increasing
- bounded below by 1



Such a sequence <u>must</u> converge to a limit: $\lim_{n\to\infty} x_n = L$, $L \geq 1$.

So $\underbrace{\lim_{n\to\infty} x_{n+1}}_{= L} = \frac{1}{2}\left(\underbrace{\lim_{n\to\infty} x_n}_{L} + \frac{1}{\underbrace{\lim_{n\to\infty} x_n}_{L}}\right) \Rightarrow L = \frac{1}{2}\left(L + \frac{1}{L}\right)$

$\Rightarrow \frac{1}{2}L = \frac{1}{2L} \Rightarrow L^2 = 1 \Rightarrow L = 1$ (since $L \geq 1$).

3) a) $f(x) = x^{-2/3}$ $\quad f(27) = \frac{1}{9}$

$f'(x) = -\frac{2}{3}x^{-5/3}$ $\quad f'(27) = -\frac{2}{3}\cdot\frac{1}{3^5} = \frac{-2}{3^6}$

$f''(x) = \frac{10}{9}x^{-8/3}$ $\quad f''(27) = \frac{10}{9}\cdot\frac{1}{3^8} = \frac{10}{3^{10}}$

$f'''(x) = -\frac{80}{27}x^{-11/3}$ $\quad f'''(27) = -\frac{80}{27}\cdot\frac{1}{3^{11}} = \frac{-80}{3^{14}}$

$$\Rightarrow \boxed{P_3(x) = \frac{1}{9} - \frac{2}{3^6}(x-27) + \frac{10}{3^{10}\cdot 2}(x-27)^2 - \frac{80}{3^{14}\cdot 6}(x-27)^3}$$

b) $$e^{x^3}\cos(x) = \left[1 + x^3 + \frac{(x^3)^2}{2} + \frac{(x^3)^3}{6} + \frac{(x^3)^4}{4!} + O(x^{15})\right] \times$$

$$\times\left[1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + O(x^{14})\right]$$

$$= 1 - \frac{1}{2}x^2 + x^3 + \frac{1}{4!}x^4 - \frac{1}{2}x^5 + \left(-\frac{1}{6!} + \frac{1}{2}\right)x^6 + \left(\frac{1}{4!} + \frac{1}{2}\right)x^7$$

$$+ \left(\frac{1}{8!} - \frac{1}{2}\right)x^6 + \left(\frac{1}{4!} + \frac{1}{2}\right)x^7 + \left(\frac{1}{8!} - \frac{1}{4}\right)x^8 + \left(-\frac{1}{6!} + \frac{1}{6}\right)x^9$$

$$+ \left(-\frac{1}{10!} + \frac{1}{2\cdot 4!}\right)x^{10} + \left(\frac{1}{8!} - \frac{1}{12}\right)x^{11} + \left(\frac{1}{12!} - \frac{1}{2\cdot 6!} + \frac{1}{4!}\right)x^{12}$$

$$+ \left(-\frac{1}{10!} + \frac{1}{4!\cdot 6}\right)x^{13} + O(x^{14})$$

$= P_{13}(x)$

c) $$\frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = 1 + (-x^5) + (-x^5)^2 + (-x^5)^3 + \cdots$$

$$= \underbrace{1 - x^5 + x^{10} - x^{15} + \cdots + (-1)^n x^{5n}}_{P_{5n}(x)} + O\left(x^{5(n+1)}\right)$$

4) $f(x)=x^{1/3}$ $\quad f'(x)=\frac{1}{3}x^{-2/3}$ $\quad f''(x)=-\frac{2}{9}x^{-5/3}$

$f(27)=3$ $\quad f'(27)=\frac{1}{27}$ $\quad f''(27)=\frac{-2}{3^7}$

So $\quad P_2(x)=3+\frac{1}{27}(x-27)-\frac{1}{3^7}(x-27)^2$

and $\quad P_2(26)=3-\frac{1}{27}-\frac{1}{3^7}$ $\quad (\approx 2.962506)$

Now $\quad f'''(x)=\frac{10}{27}x^{-8/3}$, so for $26\le s\le 27$,

$$\frac{10}{27}26^{-8/3}\ge f'''(27)\ge\frac{10}{27}27^{-8/3}$$

And so error $=\frac{f'''(s)}{3!}(-1)^3$

$$\Rightarrow -\frac{10}{27}\frac{26^{-8/3}}{3!}\le \text{error}\le -\frac{10}{27}\frac{27^{-8/3}}{3!}$$

So $(26)^{1/3}\in\left[3-\frac{1}{27}-\frac{1}{3^7}-\frac{10\cdot 26^{-8/3}}{27\cdot 3!},\ 3-\frac{1}{27}-\frac{1}{3^7}-\frac{10\cdot 27^{-8/3}}{27\cdot 3!}\right]$

$$\approx\left[\begin{matrix}2.96249531l\\ \cancel{2.962038064}\end{matrix},\ \begin{matrix}2.962496307\\ \cancel{2.962039906}\end{matrix}\right]$$

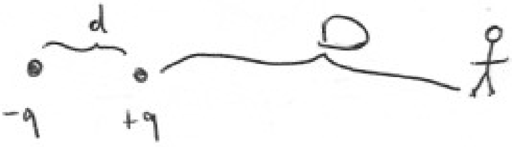
(calculator value $(26)^{1/3}\approx 2.962496086$)

5) $$E = \frac{q}{D^2} - \frac{q}{(D+d)^2} = \frac{q}{D^2}\left[1 - \frac{1}{(1+d/D)^2}\right]$$

$$= \frac{q}{D^2}\left[1-\left(\frac{1}{1-(-d/D)}\right)^2\right] = \frac{q}{D^2}\left[1-\left(1-d/D+(d/D)^2+O((d/D)^3)\right)^2\right]$$

$$= \frac{q}{D^2}\left[1-\left(1-2d/D+3(d/D)^2+O((d/D)^3)\right)\right]$$

$$= q/D^2\left[\underbrace{2d/D-3(d/D)^2}_{\text{2nd order Taylor}} + O((d/D)^3)\right]$$



$\Rightarrow$ the leading order behaviour for $d \ll D$ is

$$E \approx \frac{2qd}{D^3}$$

6) a) $$\lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} = \lim_{x\to 0} \frac{x - \frac{x^3}{6} + O(x^5) - x\left[1-x^2/2+O(x^4)\right]}{x^3}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3}x^3 + O(x^5)}{x^3} = \lim_{x\to 0}\left[\frac{1}{3}+O(x^2)\right] = \boxed{\frac{1}{3}}$$ (or use l'Hôpital)

b) $$\lim_{x\to\infty} x\left[\ln(x+5)-\ln(x)\right] = \lim_{x\to\infty} x\left[\ln[x(1+5/x)] - \ln(x)\right]$$

$$= \lim_{x\to\infty} x\left[\ln x + \ln(1+5/x) - \ln x\right] = \lim_{x\to\infty} x\ln(1+5/x)$$

$$= \lim_{x\to\infty} x\left[5/x + O\left(\frac{1}{x^2}\right)\right] = \boxed{5}$$

(or l'Hôpital $\displaystyle\lim_{x\to\infty} \frac{\ln(1+5/x)}{1/x} = \lim_{x\to\infty} \frac{-5/x^2 \frac{1}{1+5/x}}{-1/x^2} = 5$)

c) $\lim_{x\to 0} \dfrac{e^x - 2 - x + \cos(x) - x^3/6}{x^4}$

$= \lim_{x\to 0} \dfrac{\left[1 + x + x^2/2 + x^3/6 + x^4/24 + O(x^5)\right] - 2 - x + \left[1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right] - x^3/6}{x^4}$

$= \lim_{x\to 0} \dfrac{x^4/12 + O(x^5)}{x^4} = \lim_{x\to 0} \dfrac{1}{12} + O(x) = \boxed{1/12}$ (or use l'Hôpital)

d) $\ln\left(x^{\sin(x)}\right) = \sin(x)\ln x$

& $\lim_{x\to 0+} \sin(x)\ln(x) = \lim_{x\to 0+} \underbrace{\left[\frac{\sin(x)}{x}\right]}_{\to 1} \cdot \underbrace{\left[x\ln(x)\right]}_{\to 0} = 0$

$\Rightarrow \lim_{x\to 0+} x^{\sin(x)} = e^0 = \boxed{1}$

e) $\lim_{x\to 0+} (-\ln(x))^x = e^{\lim_{x\to 0+} x\ln(-\ln(x))} = e^0 = \boxed{1}$

7) $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ a) $f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \dfrac{e^{-1/h^2}}{h} = 0$

$\Rightarrow P_1(x) = 0$

b) You can show $f^{(n)}(0) = 0 \ \forall n$, hence $P_n(x) = 0 \ \forall n$!

So while $P_n(x) = 0$ may well approximate $f(x)$ for x very small, for any $x \neq 0$, $P_n(x) \not\to f(x)$ as $n \to \infty$.