Assignment 8 Solutions

1) a) $f(x) = \dfrac{2x^2}{x^2-1}$

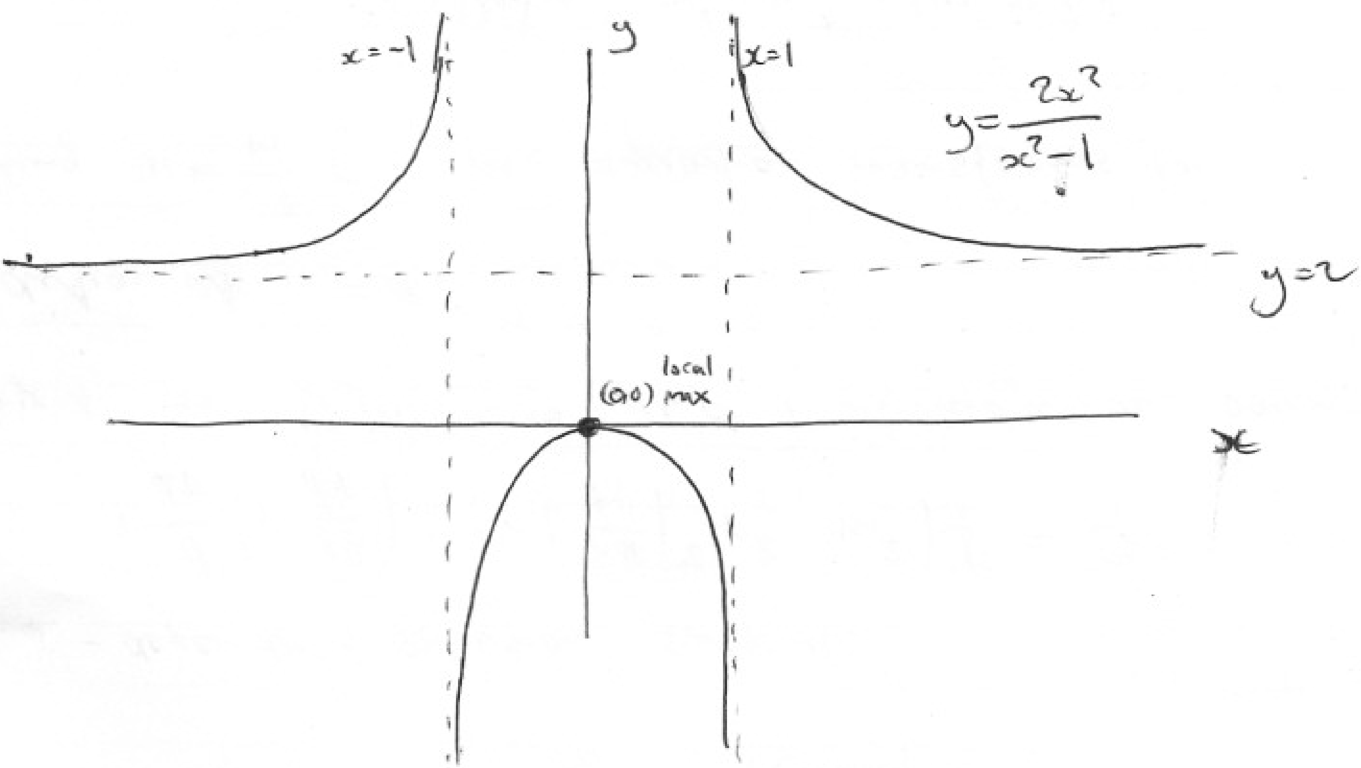
- even
- vertical asymptotes at $x = \pm 1$
- horizontal asymptote $y=2$ as $x \to \pm\infty$

- $f'(x) = \dfrac{(x^2-1)4x - 2x^2(2x)}{(x^2-1)^2} = \dfrac{4x(x^2-1-x^2)}{(x^2-1)^2} = \dfrac{-4x}{(x^2-1)^2}$

so $f'(x) = 0 \iff x = 0$, and this critical point is a local maximum ($f'>0$ for $x<0$ and $f'<0$ for $x>0$).

- $f''(x) = \dfrac{-4}{(x^2-1)^4}\left[(x^2-1)^2 - x \cdot 2(x^2-1) \cdot 2x\right] = \dfrac{-4}{(x^2-1)^3}\left[x^2-1-4x^2\right]$

$$= \frac{4(1+3x^2)}{(x^2-1)^3} \begin{cases} >0 & x^2>1 \\ <0 & x^2<1 \end{cases} \Rightarrow \begin{cases} \text{concave up} & \text{for } |x|>1 \\ \text{concave down} & \text{for } |x|<1 \end{cases}$$



b) $f(x) = 2\cos(x) + \sin(2x)$

- 2π-periodic (so it suffices to consider $0 \le x \le 2\pi$)

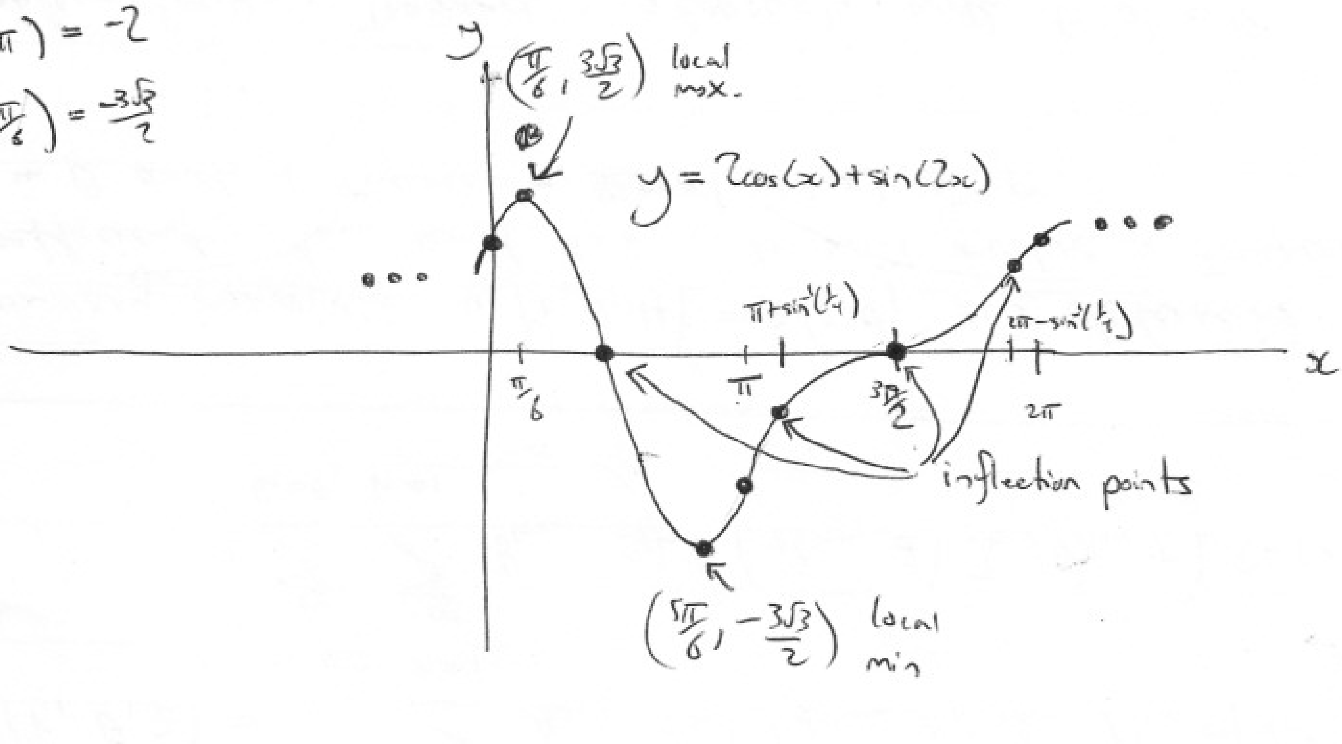
- $f'(x) = -2\sin(x) + 2\cos(2x) = -2\left(+\sin(x) - [1 - 2\sin^2(x)]\right)$
  $= -2\left[2\sin^2(x) + \sin(x) - 1\right] = -2(2\sin(x) - 1)(\sin(x) + 1)$

$\Rightarrow$ critical points: $\sin(x) = -1 \Leftrightarrow x = 3\pi/2$ (inflection, see below)

$\sin(x) = \frac{1}{2} \Leftrightarrow x = \pi/6, 5\pi/6$ ← min; $\pi/6$ ← max

- $f''(x) = -2\cos(x)[4\sin(x) + 1]$ $\begin{cases} <0 & 0 \le x < \pi/2 & \text{concave dn.} \\ >0 & \pi/2 < x < \pi + \sin^{-1}(1/4) & \text{conc. up} \\ <0 & \pi + \sin^{-1}(1/4) < x < 3\pi/2 & \text{conc. dn.} \\ >0 & 3\pi/2 < x < 2\pi - \sin^{-1}(1/4) & \text{conc. up} \\ <0 & 2\pi - \sin^{-1}(1/4) < x < 2\pi & \text{conc. dn} \end{cases}$

- $f(0) = 2$
- $f(\pi/6) = \frac{3\sqrt{3}}{2}$
- $f(\pi/2) = 0$
- $f(\pi) = -2$
- $f(5\pi/6) = \frac{-3\sqrt{3}}{2}$



c) $f(x) = \dfrac{x^3}{x^2+1}$

- odd
- oblique asymptote $y = x$ as $x \to \pm\infty$

$$\left(= \frac{x^3+x-x}{x^2+1} = x - \frac{x}{x^2+1} \right)$$

(3)

- $$f'(x) = \frac{(x^2+1)3x^2 - x^3 \cdot 2x}{(x^2+1)^2} = \frac{x^2(3x^2+3-2x^2)}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2} \geq 0$$

with $x=0$ the only critical point, neither max nor min (inflection)

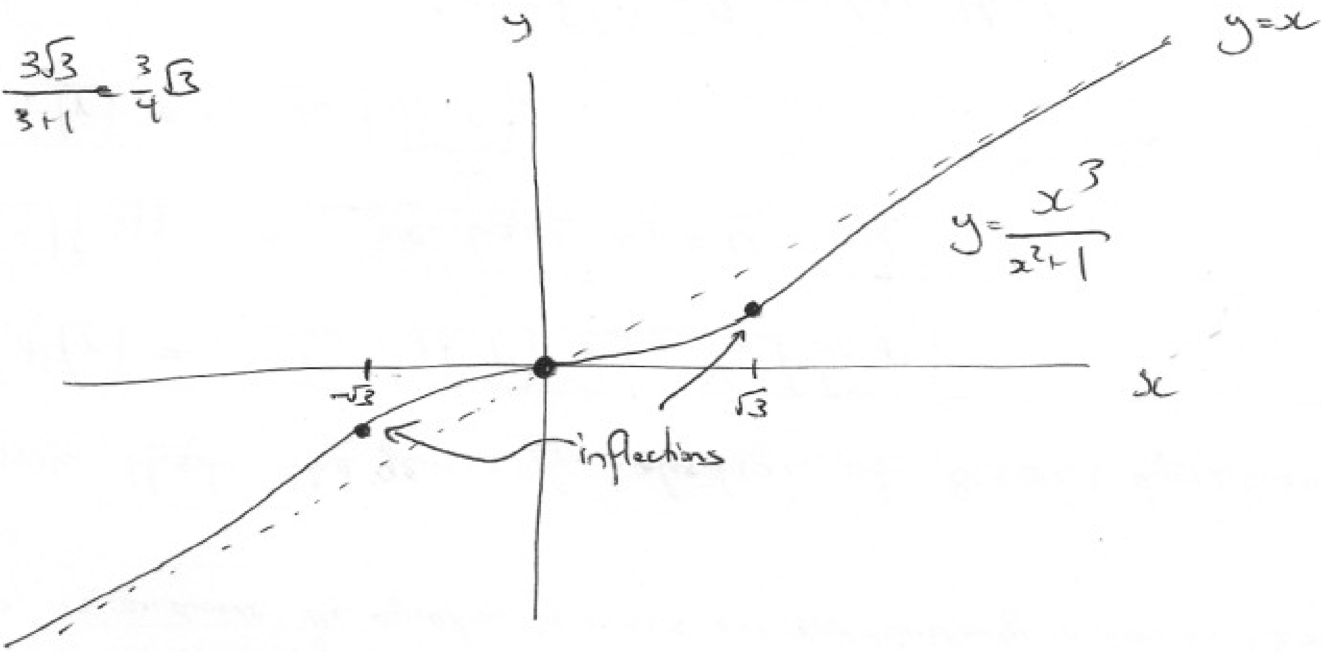
- $$f''(x) = \frac{(x^2+1)^2[4x^3+6x] - (x^4+3x^2)[2(x^2+1)2x]}{(x^2+1)^4}$$

$$= \frac{x}{(x^2+1)^3}\Big[\underbrace{(x^2+1)(4x^2+6) - 4(x^4+3x^2)}_{-2x^2+6}\Big] = \frac{2x(3-x^2)}{(x^2+1)^3}$$

$\Rightarrow$ conc. up for $x \in (-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

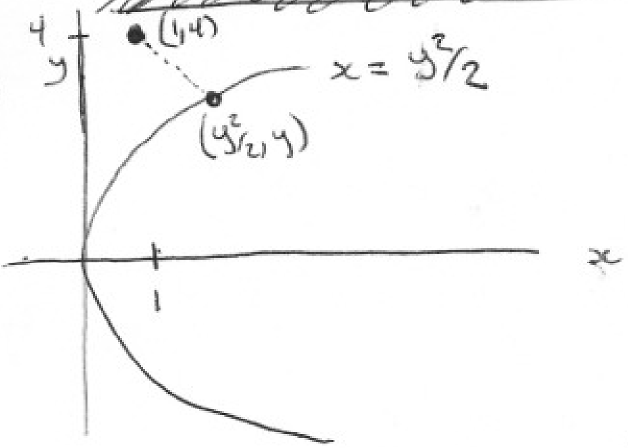
" down " $x \in (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

$f(\sqrt{3}) = \dfrac{3\sqrt{3}}{3+1} = \dfrac{3}{4}\sqrt{3}$



~~ASSIGNMENT 7 SOLUTIONS~~

2)



The distance from a point $(y^2/2, y)$ on the parabola to $(1,4)$ is

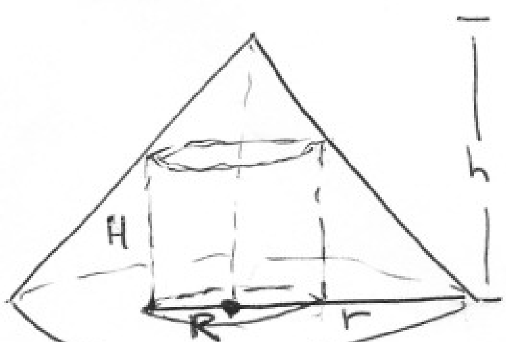
$$d(y) = \left[\left(\tfrac{y^2}{2}-1\right)^2 + (y-4)^2 \right]^{1/2}$$
$$= \left[\tfrac{1}{4}y^4 - 8y + 17 \right]^{1/2}$$

Since $\lim_{y \to \pm\infty} d(y) = +\infty$, $d(y)$ must have an absolute minimum at some y, which must be a critical point.

Solve $0 = d'(y) = \frac{1}{2}\left[\frac{y^4}{4} - 8y + 17 \right]^{-1/2} (y^3 - 8)$

$\Rightarrow y = 2 \Rightarrow$ minimum distance is $d(2) = [4 - 16 + 17]^{1/2} = \boxed{\sqrt{5}}$ to point $\boxed{(2,2)}$

3)



Let the cylinder have radius R, $0 \le R \le r$. We can find its height by similar triangles:

$$\frac{h}{r} = \frac{H}{r-R} \Rightarrow H = h\frac{r-R}{r} = h\left(1 - \frac{R}{r}\right)$$

$\Rightarrow$ the volume is $V(R) = \pi R^2 H = \pi h\left(R^2 - R^3/r\right)$.

and surface area is $A(R) = 2\pi R H = 2\pi h\left(R - R^2/r\right)$

Find critical points:

$$0 = V'(R) = \pi h\left(2R - \frac{3}{r}R^2\right) = \frac{\pi h}{r}R(2r - 3R)$$

$$\Rightarrow R = 0 \quad \text{or} \quad R = \frac{2}{3}r.$$

Since $V(0) = V(r) = 0$ and $V(\tfrac{2}{3}r) = \pi h\left(\frac{4r^2}{9} - \frac{8r^2}{27}\right)$

$$= \frac{4\pi h}{27}r^2(3-2) = \frac{4\pi}{27}hr^2$$

the maximum volume is $\boxed{\frac{4\pi}{27}hr^2}$

And

$0 = A'(R) = 2\pi h\left(1 - \frac{2R}{r}\right) \Leftrightarrow R = r/2$ and $A(0) = 0$, $A(r) = 0$ $\Rightarrow$ max. surf. area $= A(\tfrac{r}{2}) = \boxed{\frac{\pi}{4}hr}$

4) (a) $f(x) = \frac{1}{x}$ $\quad f'(x) = -\frac{1}{x^2}$ $\quad f(10) = \frac{1}{10}$ $\quad f'(10) = -\frac{1}{100}$

$$\Rightarrow \frac{1}{10.1} = f(10 + \tfrac{1}{10}) \approx f(10) + f'(10)\cdot\tfrac{1}{10} = \frac{1}{10} + \frac{1}{1000} = 0.1 + 0.001$$

$$= \boxed{0.101}$$

(b) $f(x) = \cos(x)$ $\quad f'(x) = -\sin(x)$

$f(30° = \pi/6) = \frac{\sqrt{3}}{2}$ $\quad f'(\tfrac{\pi}{6}) = -\tfrac{1}{2}$

$$\Rightarrow \cos(30.5°) = f\left(\tfrac{\pi}{6} + \tfrac{\pi}{360}\right) \approx f(\tfrac{\pi}{6}) + f'(\tfrac{\pi}{6})\tfrac{\pi}{360}$$

$$= \boxed{\frac{\sqrt{3}}{2} - \frac{\pi}{720}}$$

4) The linear approximation gives

$$f(1/2) = f(1-1/2) \approx f(1) + f'(1)(-1/2)$$
$$= 1 + 1(-1/2) = 1/2$$

The error is $\frac{1}{2} f''(s)\left(-\frac{1}{2}\right)^2 = \frac{1}{8} f''(s)$ for some $s \in \left[\frac{1}{2}, 1\right]$.

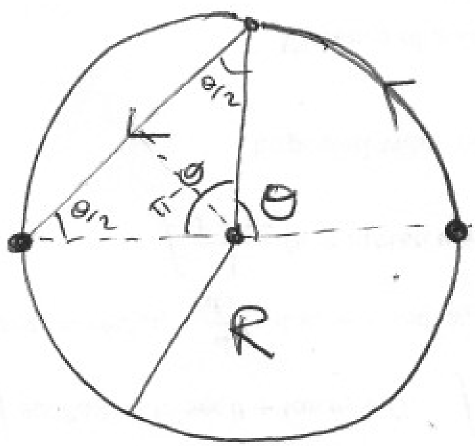
Since $f'' \geq x$ on this interval, $f'' \geq \frac{1}{2}$.

Since $f'' \leq 2-x^2$ on this interval, $f'' \leq 2-\frac{1}{4} = 7/4$

$$\Rightarrow \frac{1}{16} \leq \text{error} \leq \frac{7}{32}$$

$$\Rightarrow f\left(\tfrac{1}{2}\right) = \frac{1}{2} + \text{error}$$
$$\in \left[\frac{1}{2}+\frac{1}{16}, \frac{1}{2}+\frac{7}{32}\right]$$
$$= \boxed{\left[\frac{18}{32}, \frac{23}{32}\right]}$$

5)



Suppose you run until angle θ, then swim. So $0 \leq \theta \leq \pi$.

The running time is

$$t_{run} = \frac{\text{distance}}{\text{rate}} = \frac{R\theta}{r \cdot k}$$

k = swimming rate

$$t_{swim} = \frac{2R\cos(\theta/2)}{k}$$

$\Rightarrow$ total time is $$t(\theta) = \frac{R}{k}\left(\frac{\theta}{r} + 2\cos(\theta/2)\right)$$

Note $t(0) = R/k \cdot 2 = 2R/k$

$t(\pi) = R/k\left(\frac{\pi}{r}\right) = \frac{\pi}{r} \cdot R/k$.

Critical point?

$$0 = t'(\theta) = \frac{R}{k}\left(\frac{1}{r} - 2\sin\left(\theta/2\right) \cdot \frac{1}{2}\right)$$

$$\Rightarrow \sin\left(\theta/2\right) = 1/r$$

(a) if $r=2$, there is a critical point at $\sin\left(\frac{\theta}{2}\right) = 1/2$

$$\Rightarrow \frac{\theta}{2} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \sqrt{2}$$

and $t(\sqrt{2}) = \frac{R}{k}\left(\frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}}\right) = \frac{R}{k} \cdot 3/\sqrt{2}$

Since $3/\sqrt{2} > \frac{\pi}{2}$ and $2 > \frac{\pi}{2}$,

$\theta = \pi$ gives the minimal time $\Rightarrow$ $\boxed{\text{run the whole way}}$

(b) if $r=1$, the critical point is $\theta/2 = \frac{\pi}{2} \Rightarrow \theta = \pi$ (actually an endpoint) and $t(\pi) = \pi \cdot R/k > 2 \cdot R/k = t(0)$

$\Rightarrow \theta = 0$ gives the minimum $\Rightarrow$ $\boxed{\text{swim the whole way}}$

(c) note $t''(\theta) = -R/k \cdot \cos\left(\theta/2\right) \cdot \frac{1}{2} = \frac{-R}{4k}\cos\left(\theta/2\right)$

which is < 0 for $\theta \in (0, \pi)$ $\Rightarrow$ any critical point is a local max $\Rightarrow$ minimum is always at $\theta = 0$ or $\theta = \pi$!

$\Rightarrow$ $\boxed{\text{no}}$.

7) (a) if $ab = 1$, $b = 1/a$, so consider

$$f(a) := \frac{a^p}{p} + \frac{b^{p'}}{p'} = \frac{a^p}{p} + \frac{a^{-p'}}{p'} \quad \text{for } a > 0.$$

Note $\lim_{a \to 0+} f(a) = \lim_{a \to \infty} f(a) = \infty$.

Critical points? $0 = f'(a) = a^{p-1} - a^{-p'-1}$

$\Rightarrow a = 1$ (since $p \neq -p'$). So

$$f(a=1) = \frac{1}{p} + \frac{1}{p'} = 1 \Rightarrow \frac{a^p}{p} + \frac{b^{p'}}{p'} \geq 1 \quad \forall a>0, b>0, \ ab=1.$$

(b) now if a, b are <u>any</u> positive numbers,

$$\tilde{a} \cdot \tilde{b} = (ab)^{-1/p} a \cdot (ab)^{-1/p'} b = \frac{ab}{(ab)^{\frac{1}{p}+\frac{1}{p'}=1}} = 1$$

so part (a) applies to $\tilde{a}$ and $\tilde{b}$ to yield

$$\frac{\tilde{a}^p}{p} + \frac{\tilde{b}^{p'}}{p'} \geq 1 \Rightarrow \frac{(ab)^{-1} a^p}{p} + \frac{(ab)^{-1} b^{p'}}{p'} \geq 1$$

$$\Rightarrow \boxed{\frac{a^p}{p} + \frac{b^{p'}}{p'} \geq ab} \quad \forall a>0, b>0.$$