Assignment 7 Solutions

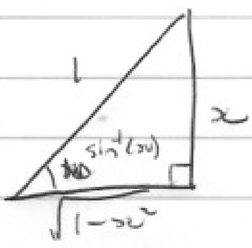
1) a) $\tan^{-1}(\tan(\frac{4\pi}{3})) = \tan^{-1}(\tan(\frac{4\pi}{3} - \pi))$

$= \tan^{-1}(\tan(-\pi/3)) = \boxed{-\pi/3}$

b) $\tan(\tan^{-1}(4\pi/3)) = \boxed{4\pi/3}$

c) $\tan(\sin^{-1}(x)) = \dfrac{\sin(\sin^{-1}(x))}{\cos(\underbrace{\sin^{-1}(x)}_{\in[-\pi/2,\,\pi/2]})} = \dfrac{x}{\sqrt{1-\sin^2(\sin^{-1}(x))}} = \boxed{\dfrac{x}{\sqrt{1-x^2}}}$

≥ 0



2) a) $f(x) = \cos^{-1}(\sin^{-1}(x))$

$f'(x) = \dfrac{-1}{\sqrt{1-[\sin^{-1}(x)]^2}} \cdot \dfrac{1}{\sqrt{1-x^2}} = \dfrac{-1}{\sqrt{1-[\sin^{-1}(x)]^2}\sqrt{1-x^2}}$

Domain of f: $\sin^{-1}(x) \in [-1, 1]$

$\Leftrightarrow \quad x \in [\sin(-1), \sin(1)]$

Domain of f': $x \in (\sin(-1), \sin(1))$

b) $g(t) = 2^{\arctan(t)}$, $\quad g'(t) = \dfrac{\ln(2)\, 2^{\arctan(t)}}{1+t^2}$

Domain of f = Domain of $f' = \mathbb{R}$

4) • ellipse $\frac{x^2}{3}+y^2=1$ and $x^2-y^2=1$ intersect

where $1-\frac{x^2}{3}=x^2-1 \iff 2=\frac{4}{3}x^2 \iff x^2=\frac{6}{4}=3/2$

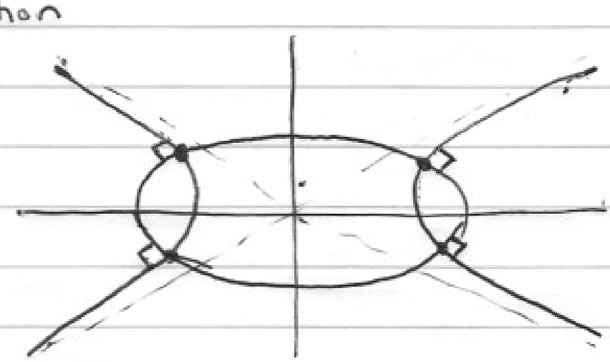
• slope of ellipse: $0=\frac{2x}{3}+2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=-\frac{x}{3y}$

• slope of hyperbola: $0=2x-2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{x}{y}$

• so if $x=\pm\frac{\sqrt{3}}{2}$,

$$(\text{ellipse slope})(\text{hyperbola slope})=-\frac{x}{3y}\cdot\frac{x}{y}=-\frac{x^2}{3y^2}=\frac{-3/2}{3(3/2-1)}=-1$$

$\Rightarrow$ perpendicular intersection



5) $\frac{1}{R}=\frac{1}{R_1}+\dots+\frac{1}{R_n} \Rightarrow -\frac{1}{R^2}\frac{dR}{dt}=-\frac{1}{R_1^2}\frac{dR_1}{dt}+\dots-\frac{1}{R_n^2}\frac{dR_n}{dt}$

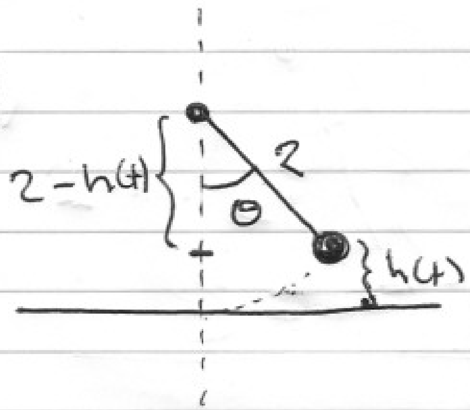
so if at $t=t_0$, $R_1(t_0)=\dots=R_n(t_0)$, then

$$\frac{dR}{dt}(t_0)=R^2(t_0)\left[\frac{1}{R_1^2(t_0)}\frac{dR_1}{dt}(t_0)+\frac{1}{R_1^2(t_0)}\frac{dR_2}{dt}(t_0)+\dots+\frac{1}{R_n^2(t_0)}\frac{dR_n}{dt}(t_0)\right]$$

$$=\frac{R^2(t_0)}{R_1^2(t_0)}\left[\frac{dR_1}{dt}(t_0)+\dots+\frac{dR_n}{dt}(t_0)\right]=0$$

if $\boxed{\frac{dR_1}{dt}(t_0)+\dots+\frac{dR_n}{dt}(t_0)=0}$

6)

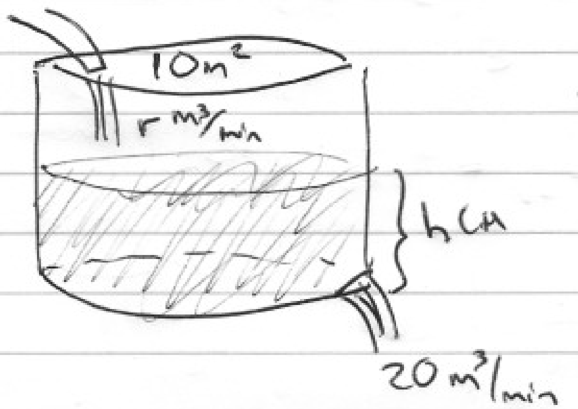


$$2 - h(t) = 2\cos\theta(t)$$

$$\Rightarrow -\frac{dh}{dt} = -2\sin\theta \frac{d\theta}{dt}$$

$$\Rightarrow \left.\frac{dh}{dt}\right|_{\substack{\theta = \pi/4 \\ d\theta/dt = 1/2}} = 2 \cdot \sin(\pi/4) \cdot 1/2 = \boxed{\frac{1}{\sqrt{2}} \text{ m/s.}}$$

7)



volume $V(t) = 10\,h(t)$

$$\Rightarrow \frac{dV}{dt} = 10\frac{dh}{dt}$$

$\frac{dV}{dt}$ = rate in − rate out $= r - 20$

$$\Rightarrow r = 20 + 10 \cdot 2 = \boxed{40 \text{ m}^3/\text{min}}$$

8) $f(x) = x^{2/3} - x^{8/3}$ $\quad |x| \le 2$

$f'(x) = \frac{2}{3}x^{-1/3} - \frac{8}{3}x^{5/3} = \frac{2}{3}x^{-1/3}(1 - 4x^2)$ $\quad 0 < |x| \le 2$

$\Rightarrow$ critical points : $x = \pm 1/2$ $\quad (\in [-2,2])$

singular point : $x = 0$

endpoints $\quad x = \pm 2$

$f(\pm 2) = 2^{2/3} - 2^{8/3} = 2^{2/3}(1 - 2^2) = -3 \cdot 2^{2/3}$ $\quad$ <u>abs. min.</u>

$f(\pm 1/2) = 2^{-2/3} - 2^{-8/3} = 2^{-8/3}(2^2 - 1) = 3 \cdot 2^{-8/3}$ $\quad$ <u>abs. max.</u>

$f(0) = 0$ is a <u>local min.</u>, since f is increasing to the right of $x=0$, and decreasing to the left of $x = 0$.

9) $$f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

a) • f is cont. for $x \neq 0$, since $x^2\sin(\frac{1}{x})$ is

• $\lim_{x\to 0} x^2\sin(\frac{1}{x}) = 0 = f(0)$ by the squeeze theorem

$\left(-x^2 \leq x^2\sin(\frac{1}{x}) \leq x^2 \quad \forall x \neq 0\right) \Rightarrow f$ cont. at $x=0$ also.

• set $a_n = \dfrac{1}{\pi/2 + 2\pi n}$, $b_n = \dfrac{1}{-\pi/2 + 2\pi n}$, $n \in \mathbb{Z}^+$

and note $f(a_n) = a_n^2 \sin(\pi/2 + 2\pi n) = a_n^2 > 0$
$f(b_n) = b_n^2 \sin(-\pi/2 + 2\pi n) = -b_n^2 < 0$

and $a_n, b_n \longrightarrow 0$ as $n \to \infty$.

Thus $f(0)=0$ is not a local max. or min. value.

b) • f is diff. for $x \neq 0$, since $x^2\sin(\frac{1}{x})$ is

• $f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0} h \cdot \sin(\frac{1}{h}) = 0$

(squeeze theorem again) $\Rightarrow f$ diff. at $x=0$ also.

• we already know $f(0)$ is neither a local max. or min. value

• for $x > 0$, $f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$
$f''(x) = 2\sin(\frac{1}{x}) - \frac{2}{x}\cos(\frac{1}{x}) + \frac{1}{x^2}\sin(\frac{1}{x})$

so for n large $f''(a_n) \approx \frac{1}{a_n^2} > 0$, $f''(b_n) \approx \frac{-1}{b_n^2} < 0$

$\Rightarrow$ there is no interval $(0,h)$ (no matter how small h) on which f is concave up or concave down

$\Rightarrow$ $x=0$ is not an inflection point.