Assignment 6 Solutions

1) a) $\frac{d}{dx} 3^{x^2} = \boxed{3^{x^2} \ln(3) \cdot 2x}$

b) $\frac{d}{dx} e^{-2x} \ln\left(\frac{1}{x}\right) = -2e^{-2x} \ln\left(\frac{1}{x}\right) + e^{-2x} \cdot \frac{1}{1/x} \cdot \frac{-1}{x^2}$

$= \boxed{-e^{-2x}\left(2\ln\left(\frac{1}{x}\right) + \frac{1}{x}\right)}$

c) $\ln(e^{t^2} e^{-t}) = \ln(e^{t^2-t}) = t^2 - t$

$\Rightarrow \frac{d}{dx} \ln(e^{t^2}e^{-t}) = \boxed{2t-1}$

d) $\frac{d}{dx} \ln(\ln(\ln(x))) = \boxed{\frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}}$

e) $\frac{d}{dx} \coth(x) = \frac{d}{dx} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$

$= \boxed{\frac{-4}{(e^x - e^{-x})^2}}$ $\left(= -\left[\frac{2}{e^x - e^{-x}}\right]^2 = -\frac{1}{\cosh^2(x)} = -\operatorname{csch}^2(x) \right)$

f) $\frac{d}{dx} x^{x^3}$. $f(x) := x^{x^3}$, $\ln f(x) = x^3 \ln(x)$.

$\Rightarrow \frac{f'(x)}{f(x)} = \frac{d}{dx}\left[x^3 \ln(x)\right] = x^3 \cdot \frac{1}{x} + 3x^2 \cdot \ln(x)$

$\Rightarrow f'(x) = \boxed{x^{x^3}(x^2 + 3x^2 \ln(x))}$

2) (Remark: this question should <u>not</u> have appeared)

Differentiate $xye^{x/y} + \frac{1}{e} = 0$ implicitly, to get

$$ye^{x/y} + x\frac{dy}{dx}e^{x/y} + xye^{x/y}\left[\frac{-x}{y^2}\frac{dy}{dx} + \frac{1}{y}\right] = 0$$

$$\Rightarrow e^{x/y}\left[\left(x - x^2/y\right)\frac{dy}{dx} + y + x\right] = 0$$

$$\Rightarrow \left(x^2/y - x\right)\frac{dy}{dx} = x + y$$

At $x=1, y=-1$, $(-1-1)\frac{dy}{dx} = 1-1 = 0 \Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow$ the tangent line is $\boxed{y=-1}$

~~3) T(t) = temperature of water~~

3) $T(t)$ = temperature of water at time t

$T(0) = 0, \quad T(10) = 5$

Newton's law of cooling: $T' = k(T - \underset{\text{room temp.}}{20})$

Set $y(t) = T(t) - 20$, so

$$\frac{dy}{dt} = T'(t) = ky$$

$$y(0) = T(0) - 20 = -20$$

$$\Rightarrow y(t) = -20e^{kt}$$

Since $T(10) = 5$, $\quad y(10) = T(10) - 20 = 5 - 20 = -15$

$$\Rightarrow -15 = -20e^{k \cdot 10} \Rightarrow e^{10k} = 3/4 \Rightarrow 10k = \ln(3/4)$$
$$\Rightarrow k = \frac{1}{10}\ln(3/4).$$

So $\quad T(t) = 20 + y(t) = 20\left(1 - e^{\frac{1}{10}\ln(3/4)t}\right)$,

and $T(t) = 10 \Leftrightarrow 1 - e^{\frac{1}{10}\ln(3/4)t} = 1/2$

$$\Leftrightarrow e^{\frac{1}{10}\ln(3/4)t} = \frac{1}{2} \Leftrightarrow \frac{1}{10}\ln(3/4)t = \ln\left(\frac{1}{2}\right)$$

$$\Leftrightarrow t = \frac{10\ln\left(\frac{1}{2}\right)}{\ln(3/4)} \boxed{= \frac{10\ln(2)}{\ln(4/3)} \approx 24 \text{ min.}}$$

It will take infinite time to reach temperature 20. In fact,

$$\boxed{\lim_{t \to \infty} T(t) = 20}$$

4) If $f'(x) = f(x)$ for all x, then

$$\frac{d}{dx}\left[e^{-x}f(x)\right] = -e^{-x}f(x) + e^{-x}f'(x)$$
$$= e^{-x}\left[f'(x) - f(x)\right] = 0 \quad \forall x,$$

and so $e^{-x}f(x) = C$ (a constant), hence
$f(x) = Ce^{x}$.

5) $f(x) := x^{x} \Rightarrow \ln f(x) = x\ln(x)$
and we know $\lim_{x\to 0+} \ln f(x) = \lim_{x\to 0+} x\ln(x) = 0$

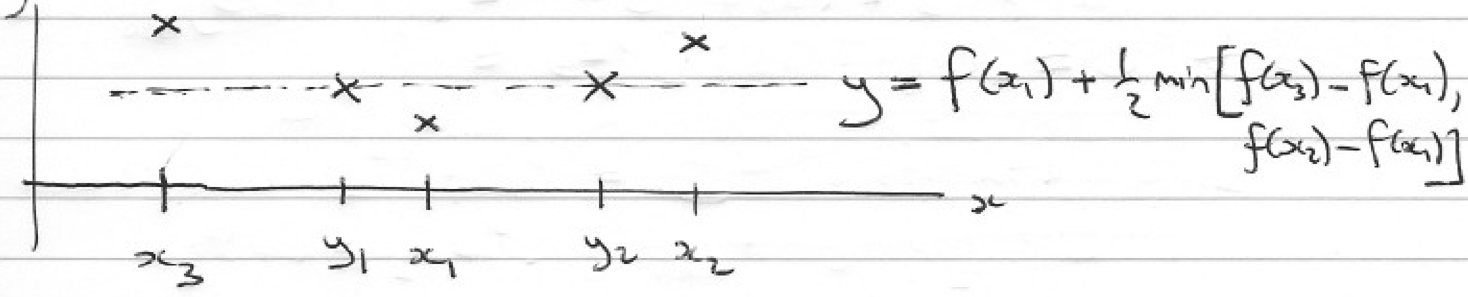
$$\lim_{x\to 0+} \ln f(x) = \ln\left[\lim_{x\to 0+} f(x)\right] \Rightarrow \boxed{\lim_{x\to 0+} x^{x} = 1}$$

$\therefore$ then $\lim_{x\to 0+} x^{x^{x}} = \left(\lim_{x\to 0+} x\right)^{\left(\lim_{x\to 0+} x^{x}\right)} = 0^{1} = \boxed{0}$.

6) Let f be continuous and 1-1 on $\mathbb{R}$.
Let $x_1 < x_2$ be any 2 points. (since f is 1-1)
We cannot have $f(x_1) = f(x_2)$ so suppose $f(x_1) < f(x_2)$, and we will show f is increasing (if $f(x_1) > f(x_2)$, a similar argument shows f is decreasing).

If $x_3 < x_1$, and if $f(x_3) > f(x_1)$. Then by the intermediate value theorem (f is continuous), there are $y_1 \in (x_3, x_1)$ and $y_2 \in (x_1, x_2)$ such that
$$f(y_1) = f(y_2) = f(x_1) + \frac{1}{2}\min\left[f(x_3) - f(x_1), f(x_2) - f(x_1)\right]$$
(see picture) which is a contradiction to f being 1-1.
So we must have $f(x_3) < f(x_1)$, so f is increasing on $[-\infty, x_1]$.

One can show f is increasing on $[x_1, x_2]$ and $[x_2, \infty)$ in a very similar way.



7) For $h > 0$, apply MVT on $[a, a+h]$
(since f cont. on $[a, a+h]$, diff. on $(a, a+h)$)
so $\exists$ $c = c(h) \in (a, a+h)$ such that

$$\frac{f(a+h) - f(a)}{h} = f'(c(h)).$$

By the squeeze theorem, since $a < c(h) < a+h$, $\lim_{h \to 0} c(h) = a$. So

$$\lim_{h \to 0+} \frac{f(a+h) - f(a)}{h} = \lim_{c \to a} f'(c) \text{ exists.}$$

Similarly $\lim_{h \to 0-} \frac{f(a+h) - f(a)}{h} = \lim_{c \to a} f'(c)$.

Hence f is differentiable at a, with $f'(a) = \lim_{c \to a} f'(c)$.