Assignment 5 Solutions

1) a) $y' = x^2 - 1/x^3$

$\Rightarrow \quad y = \frac{1}{3}x^3 + \frac{1}{2x^2} + C$

& $y(2) = 2 \Rightarrow \quad 2 = \frac{1}{3}2^3 + \frac{1}{2\cdot 2^2} + C = \frac{8}{3} + \frac{1}{8} + C$

$\Rightarrow C = \frac{1}{24}\left[48 - 64 - 3\right] = -19/24$

$$\Rightarrow \boxed{y(x) = \frac{1}{3}x^3 + \frac{1}{2x^2} - \frac{19}{24}}$$

b) $y'' = 2\sin(3x), \quad y(0) = 1, \quad y'(0) = 0$

$\Rightarrow \quad y' = -\frac{2}{3}\cos(3x) + C$

$0 = y'(0) = -2/3 + C \quad \Rightarrow \quad C = 2/3$

$\Rightarrow \quad y = -\frac{2}{9}\sin(3x) + \frac{2}{3}x + D$

$1 = y(0) = D \quad \Rightarrow$

$$\boxed{y = -\frac{2}{9}\sin(3x) + \frac{2}{3}x + 1}$$

2) Let $t=0$ be the time when the moose is first spotted. For $0 \le t \le \frac{1}{2}$, we have

$v(t) = 30\ m/s$, $a(t) = 0$, and

$x(t) = 30t + C$. If we let $x=0$ be the car location at time $t=0$, then $x(0)=0$ so

$0 = 30 \cdot 0 + C = C \Rightarrow C = 0$.

For $t \ge \frac{1}{2}$, (until stopping) we have

$a(t) = -10\ m/s^2$. So

$v(t) = -10t + D$. Since $v(\frac{1}{2}) = 30$, we see $30 = -10 \cdot \frac{1}{2} + D \Rightarrow D = 30 + 5 = 35$

$\Rightarrow v(t) = 35 - 10t$.

So $x(t) = 35t - 5t^2 + E$. Since $x(\frac{1}{2}) = 30 \cdot \frac{1}{2} = 15$,

we get $15 = 35 \cdot \frac{1}{2} - 5 \cdot \frac{1}{4} + E$

$\Rightarrow E = 15 - \frac{35}{2} + \frac{5}{4} = \dfrac{60 - 70 + 5}{4} = -5/4$.

The car stops when $v(t) = 0 \Leftrightarrow t = 7/2$. Summary:

$$a(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} \\ -10 & \frac{1}{2} < t \le \frac{7}{2} \end{cases} \quad v(t) = \begin{cases} 30 & 0 \le t \le \frac{1}{2} \\ 35 - 10t & \frac{1}{2} < t \le \frac{7}{2} \end{cases}$$

$$x(t) = \begin{cases} 30t & 0 \le t \le \frac{1}{2} \\ 35t - 5t^2 - 5/4 & \frac{1}{2} < t \le \frac{7}{2} \end{cases}$$

The car travels $x(7/2) = \frac{35\cdot 7\cdot 2 - 5\cdot 7^2 - 5}{4}$

$$= \frac{5(\overbrace{7^2\cdot 2 - 7^2 - 1}^{48})}{4} = \boxed{60\text{ m}}$$ before stopping.

The moose lives! (for now).

3) $f(x) = x^4 + x^2 + 1$

For $x \geq 0$, $f'(x) = 4x^3 + 2x > 0$, so f is increasing on $[0,\infty) \Rightarrow f$ 1-1 on $[0,\infty)$.

If $f^{-1}(y) = x$, then $y = f(x) = x^4 + x^2 + 1$

$$\Rightarrow (x^2)^2 + (x^2) + 1 - y = 0$$

since $x^2 \geq 0$

$\Rightarrow$ (quadratic formula) $$x^2 = \frac{-1 \pm \sqrt{1+4(y-1)}}{2} = -\frac{1}{2} + \frac{1}{2}\sqrt{4y-3}$$

since $x \geq 0$

and so $$x = \pm\sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{4y-3}} \Rightarrow \boxed{f^{-1}(y) = \sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{4y-3}}}$$

with $\text{domain}(f^{-1}) = \text{range}(f) = \boxed{[1,\infty)}$

4) $f(x) = x^3 + 3x^2 + 4x + 1$ ④

$f'(x) = 3x^2 + 6x + 4$

$= 3(x^2 + 2x) + 4$ (completing the □)

$= 3((x+1)^2 - 1) + 4 = 3(x+1)^2 + 1 > 0$

$\Rightarrow$ f is increasing $\forall x$ $\Rightarrow$ f is one-to-one.

$f(1) = 1 + 3 + 4 + 1 = 9$, so $f^{-1}(9) = 1$

and $(f^{-1})'(9) = \dfrac{1}{f'(1)} = \dfrac{1}{3+6+4} = \boxed{\dfrac{1}{13}}$

5) • for $x > 0$, $f(x) = x^2 \Rightarrow f'(x) = 2x > 0 \Rightarrow$ increasing

• for $x < 0$, $f(x) = -x^2 \Rightarrow f'(x) = -2x > 0 \Rightarrow$ increasing

• note $f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h} = \lim_{h \to 0} \dfrac{h^2}{|h|} = \lim_{h \to 0} |h| = 0$

so f is differentiable $\forall x$ (and in particular continuous).

• Thus f is increasing on $(-\infty, 0]$ and $[0, \infty)$, and hence on all $\mathbb{R}$

and so f is one-to-one.

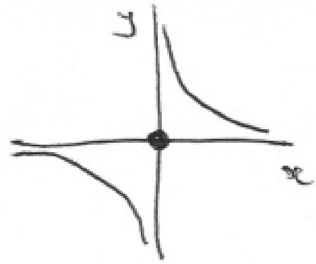
• solve $y = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases} \Rightarrow$ $x = \sqrt{y}$ if $y \geq 0$; $x = -\sqrt{-y}$ if $y \leq 0$

$\Rightarrow \boxed{f^{-1}(y) = \begin{cases} \sqrt{y} & y \geq 0 \\ -\sqrt{-y} & y < 0 \end{cases}}$

f^{-1} is differentiable at ~~wherever~~ if f is diff. at $f^{-1}(y)$ and $f'(f^{-1}(y)) \neq 0$

i.e. at $\boxed{y \neq 0}$

6) The functions $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



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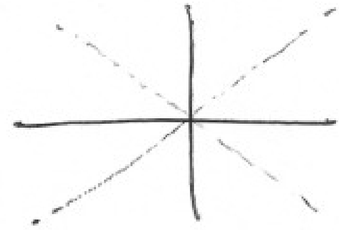
or $f(x) = \begin{cases} x & x \text{ rational} \\ -x & \text{irrational} \end{cases}$



are both 1–1 on $\mathbb{R}$, but neither decreasing nor increasing on $\mathbb{R}$ (note the 2nd example does not increase or decrease on any interval).

There are many other possible examples, all of them discontinuous.