Assignment 4 Solutions

1) a) $\frac{d}{dt}\cos(\sec(t)) = -\sin(\sec(t))\sec(t)\tan(t)$

b) $$\frac{d}{dx}\left[\csc^2(x)-\cot^2(x)\right] = 2\csc(x)\cdot(-\csc(x)\cot(x)) - 2\cot(x)\cdot(-\csc^2(x)) = 0,$$

which is not surprising since

$$\csc^2(x)-\cot^2(x) = \frac{1}{\sin^2(x)} - \frac{\cos^2(x)}{\sin^2(x)} = \frac{-\sin^2(x)}{\sin^2(x)} = -1, \text{ a constant}$$

c)
$$f(x) = \sec(x)$$
$$f'(x) = \sec(x)\tan(x)$$
$$\begin{aligned} f''(x) &= \sec(x)\cdot\sec^2(x) + \sec(x)\tan(x)\cdot\tan(x) \\ &= \sec(x)\left[\sec^2(x)+\tan^2(x)\right] = \sec(x)\left[2\sec^2(x)-1\right] \\ &= 2\sec^3(x) - \sec(x) \end{aligned}$$
$$f'''(x) = (6\sec^2(x)-1)\sec(x)\tan(x)$$

d) $f(x) = \sin(kx)$, $f'(x) = k\cos(kx)$, $f''(x) = -k^2\sin(kx)$,
$f'''(x) = -k^3\cos(kx)$, $f^{(4)}(x) = k^4\sin(kx)$, ...

$$\Rightarrow \quad f^{(n)}(x) = \begin{cases} (-1)^{n/2}k^n\sin(kx) & n \text{ even} \\ (-1)^{\frac{n-1}{2}}k^n\cos(kx) & n \text{ odd} \end{cases}$$

e) $f = (1-3x)^{-1/2}$ $\quad f' = -\frac{1}{2}(1-3x)^{-3/2}(-3) = \frac{3}{2}(1-3x)^{-3/2}$

$$f'' = \frac{3}{2}\cdot -\frac{3}{2}(1-3x)^{-5/2}\cdot(-3) = 3^2\cdot\frac{1}{2}\cdot\frac{3}{2}(1-3x)^{-5/2}$$

$$f''' = 3^2\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\left(-\frac{5}{2}\right)(1-3x)^{-7/2}(-3) = 3^3\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}(1-3x)^{-7/2}$$

$$\Rightarrow f^{(n)}(x) = \left(\frac{3}{2}\right)^n (2n-1)\cdot(2n-3)\cdot\ldots\cdot(1)\,(1-3x)^{-(2n+1)/2}$$

$$= \left(\frac{3}{2}\right)^n \frac{(2n-1)!}{(2n-2)(2n-4)\cdots 2}(1-3x)^{-(2n+1)/2}$$

$$= \left(\frac{3}{2}\right)^n \frac{(2n-1)!}{2^{n-1}(n-1)!}(1-3x)^{-(2n+1)/2}$$

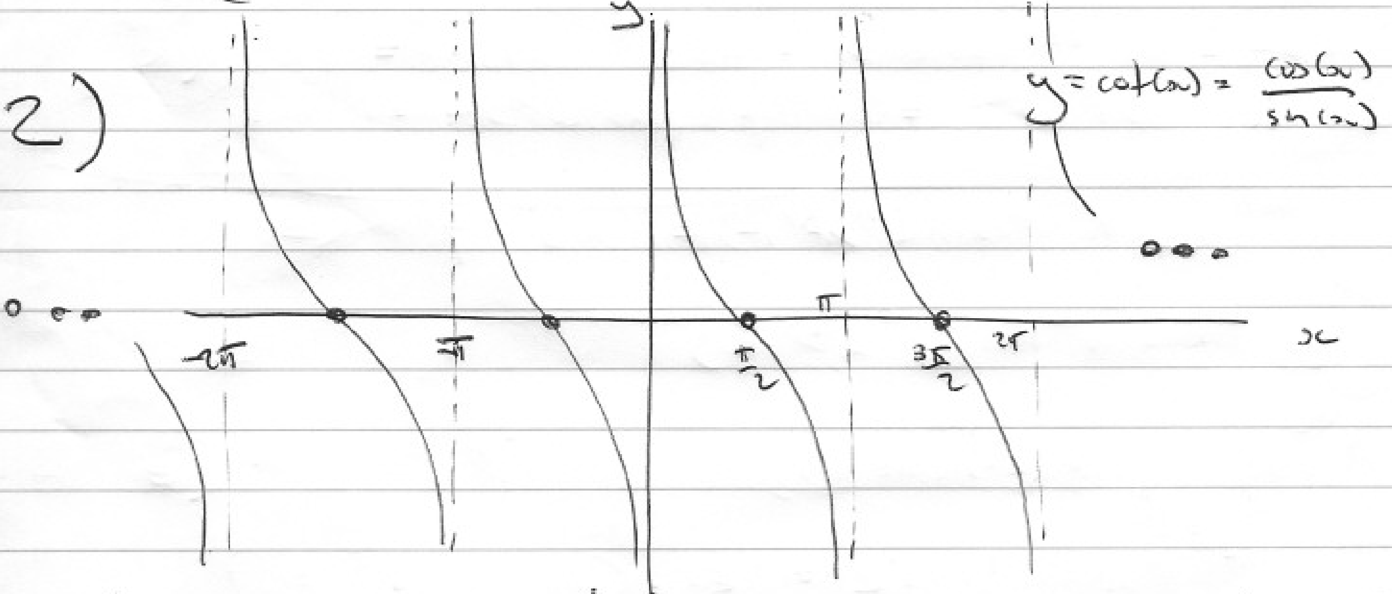
$$= \boxed{2\left(\frac{3}{4}\right)^n \frac{(2n-1)!}{(n-1)!}(1-3x)^{-(n+1/2)}}$$

f) since $\cos(x) = \cos(2\cdot \frac{x}{2}) = 1 - 2\sin^2(\frac{x}{2})$

$$\lim_{x\to 0} \frac{1-\cos x - x^2/2}{x^2} = \lim_{x\to 0}\frac{2\sin^2(x/2) - x^2/2}{x^2} = \lim_{x\to 0}\frac{1}{2}\left(\left[\frac{\sin(x/2)}{x/2}\right]^2 - 1\right)$$

$$= \frac{1}{2}\left(\left[\lim_{x\to 0}\frac{\sin(x/2)}{x/2}\right]^2 - \lim_{x\to 0} 1\right) = \frac{1}{2}(1-1) = \boxed{0}$$

2)



- $\frac{dy}{dx} = -\csc^2(x) = -\frac{1}{\sin^2(x)}$ never $=0$ $\Rightarrow$ no horizontal tangents
- $\frac{dy}{dx} = -2 \iff -\frac{1}{\sin^2(x)} = -2 \iff \sin^2(x) = \frac{1}{2} \iff \sin(x) = \pm\frac{1}{\sqrt{2}}$

$$\iff \boxed{x = \pi/4 + j\pi/2 \quad j = 0, \pm 1, \pm 2, \ldots}$$

3) $f(x) = x^3 - 4x + 1$

$$f'(x) = 3x^2 - 4 = 3(x^2 - 4/3)$$
$$= 3(x - 2/\sqrt{3})(x + 2/\sqrt{3})$$

$$\Rightarrow f'(x) \begin{cases} > 0 & \text{if } x < -2/\sqrt{3} \text{ or } x > 2/\sqrt{3} \\ < 0 & \text{if } -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}} \end{cases}$$

$\Rightarrow$ f is increasing on $(-\infty, -2/\sqrt{3}]$
and $[2/\sqrt{3}, \infty)$

and decreasing on $[-2/\sqrt{3}, 2/\sqrt{3}]$

4) $f(x) = x^{2/3}$, $f'(x) = \frac{2}{3}x^{-1/3}$

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - 1}{2} = 0,$$ and $f'(x)$ is <u>never</u> 0.

The MVT does <u>not</u> apply to f on $[-1, 1]$ since f is <u>not</u> differentiable in $(-1, 1)$
(in particular at $x = 0$).

5). • ball

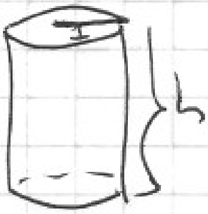
$$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow r = \left(\frac{3}{4\pi}\right)^{1/3} V^{\frac{1}{3}}$$

$$\Rightarrow \frac{dr}{dV} = \left(\frac{3}{4\pi}\right)^{1/3} \frac{1}{3} V^{-\frac{2}{3}}$$

$$\Rightarrow \left.\frac{dr}{dV}\right|_{V=1} = \left(\frac{3}{4\pi}\right)^{1/3} \frac{1}{3} = \left(\frac{1}{36\pi}\right)^{\frac{1}{3}}$$

• cylinder



$$V = \pi h \Rightarrow h = \frac{1}{\pi} V$$

$$\Rightarrow \frac{dh}{dV} = \frac{1}{\pi}$$

• note $\left(\frac{1}{\pi}\right)^3 \simeq \frac{1}{27}$ while $\left(\left(\frac{1}{36\pi}\right)^{\frac{1}{3}}\right)^3 = \frac{1}{36\pi} \simeq \frac{1}{108}$

$\Rightarrow$ <u>h is changing faster.</u>

6) Let $x_1 < x_2 < x_3 < x_4$ be points where f is 0.
Applying the MVT on $[x_1, x_2], [x_2, x_3], [x_3, x_4]$, to yield $c_1 \in (x_1, x_2)$, $c_2 \in (x_2, x_3)$, $c_3 \in (x_3, x_4)$ with $f'(c_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$, $f'(c_2) = 0$, $f'(c_3) = 0$.
Apply MVT again for $f'(x)$ on $[c_1, c_2]$ and $[c_2, c_3]$ to get $b_1 \in (c_1, c_2)$ and $b_2 \in (c_2, c_3)$ with $f''(b_1) = f''(b_2)$.
Finally MVT for f'' on $[b_1, b_2]$ yields $y \in (b_1, b_2)$ with $f'''(y) = 0$.