Assignment 3 Solutions, Math 120

1) $\left[\left(4-x^{2/5}\right)^{-5/2}\right]' = -\frac{5}{2}\left(4-x^{2/5}\right)^{-7/2} \cdot -\frac{2}{5}x^{-3/5} = \boxed{\left(4-x^{2/5}\right)^{-7/2} x^{-3/5}}$

differentiable for $x \in (-32,0) \cup (0,32)$

2) $\frac{d}{dt}\left(\frac{\sqrt{1+t^2}-1}{\sqrt{1+t^2}+1}\right) = \frac{\left(\sqrt{1+t^2}+1\right)\frac{1}{2}(1+t^2)^{-1/2}2t - \left(\sqrt{1+t^2}-1\right)\frac{1}{2}(1+t^2)^{-1/2}2t}{\left(\sqrt{1+t^2}+1\right)^2}$

$= \boxed{\frac{2t}{\sqrt{1+t^2}\left(\sqrt{1+t^2}+1\right)^2}}$ differentiable for all t

3) $\left[f\left(\left(g(xf(x))\right)^2\right)\right]'$

$= \boxed{f'\left(\left(g(xf(x))\right)^2\right) \cdot 2g(xf(x))\,g'(xf(x)) \cdot \left(f(x)+xf'(x)\right)}$

4) $\frac{d}{dx}x^{-2} = \lim_{h\to 0}\frac{(x+h)^{-2}-x^{-2}}{h} = \lim_{h\to 0}\frac{1}{h}\frac{x^2-(x+h)^2}{(x+h)^2x^2}$

$= \lim_{h\to 0}\frac{1}{h}\frac{(-2xh-h^2)}{(x+h)^2x^2} = \lim_{h\to 0}\frac{-(2x+h)}{(x+h)^2x^2} = -\frac{2}{x^3} \quad (x \neq 0)$

5) $\frac{dy}{dx} = \frac{\left(x^{1/3}+\sqrt{x}\right)(2x) - (x^2+3)\left(\frac{1}{3}x^{-2/3}+\frac{1}{2}x^{-1/2}\right)}{\left(x^{1/3}+\sqrt{x}\right)^2}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = \frac{2\cdot 2 - 4\left(\frac{1}{3}+\frac{1}{2}\right)}{2^2} = +1/6$

$x=1 \Rightarrow y = \frac{4}{2} = 2$

$\Rightarrow$ tangent line is $y = \frac{1}{6}(x-1)+2 \Rightarrow \boxed{y = \frac{1}{6}x + 11/6}$

6) $y = x^2$ and $y = \frac{1}{\sqrt{x}}$ intersect where

$$x^2 = 1/\sqrt{x} \iff x^{5/2} = 1 \iff x = 1$$

$$\frac{dx^2}{dx}\Big|_{x=1} = 2x\Big|_{x=1} = 2 \quad \& \quad \frac{d}{dx}\frac{1}{\sqrt{x}}\Big|_{x=1} = \frac{d}{dx}x^{-1/2}\Big|_{x=1} = -\frac{1}{2}x^{-3/2}\Big|_{x=1} = -\frac{1}{2}$$

& since 2 & $-\frac{1}{2}$ are negative reciprocals, the tangent lines to the 2 graphs at $(1,1)$ are orthogonal.

7)
$$\lim_{x \to 0} f_j(x) = \lim_{x \to 0} \frac{x^j}{|x|} = \begin{cases} \lim_{x\to 0} \frac{x}{|x|} \text{ does not exist} & j=1 \\ \lim_{x\to 0} \frac{x^2}{|x|} = \lim_{x\to 0} |x| = 0 & j=2 \\ \lim_{x\to 0} \frac{x^3}{|x|} = \lim_{x\to 0} |x|x = 0 & j=3. \end{cases}$$

So since $f(0) = 0$,

$\boxed{f_2, f_3 \text{ are continuous at } 0,\ f_1 \text{ is not}}$

$$\lim_{h \to 0} \frac{f_j(0+h) - f_j(0)}{h} = \lim_{h\to 0} \frac{h^j}{|h|h} = \lim_{h \to 0} \frac{h^{j-1}}{|h|}$$

$$= \begin{cases} \lim_{h\to 0} \frac{1}{|h|} \text{ does not exist} & j=1 \\ \lim_{h\to 0} \frac{h}{|h|} \text{ " " "} & j=2 \\ \lim_{h\to 0} \frac{h^2}{|h|} = \lim_{h\to 0} |h| = 0 & j=3 \end{cases}$$

So $\boxed{f_3 \text{ is differentiable at } 0, \text{ while } f_1 \text{ and } f_2 \text{ are not.}}$