Assignment # 2 Solutions

1) a) $\lim_{x\to 64}\left(x^{1/3}+3\sqrt{x}\right) = 64^{1/3}+3\sqrt{64} = 4+3\cdot 8 = \boxed{28}$

b) $\lim_{h\to 0}\dfrac{(2+h)^{-2}-1/4}{h} = \lim_{h\to 0}\dfrac{1}{h}\,\dfrac{1-\frac14(2+h)^2}{(2+h)^2} = \lim_{h\to 0}\dfrac{-h-\frac14 h^2}{h(2+h)^2} = \lim_{h\to 0}\dfrac{-1-h/4}{(2+h)^2}$

$= \boxed{-1/4}$

c) for $x\in(-2,1)$, $x^2+x-2 = (x+2)(x-1) < 0$

hence $\lim_{x\to 1^-}\sqrt{x^2+x-2}$ $\boxed{\text{does not exist}}$

d) $\dfrac{x^3-\sqrt{x}}{1-\sqrt{x}} = \dfrac{\sqrt{x}}{1-\sqrt{x}}\left(x^{5/2}-1\right) = \dfrac{\sqrt{x}}{1-\sqrt{x}}\left(x^{1/2}-1\right)\left(x^2+x^{3/2}+x+x^{1/2}+1\right)$

$= -x^{1/2}\left(x^2+x^{3/2}+x+x^{1/2}+1\right)$ for $x\neq 1$, so

$\lim_{x\to 1}\dfrac{x^3-\sqrt{x}}{1-\sqrt{x}} = -1(1+1+1+1+1) = \cancel{-4} = \boxed{-5}$

e) for $x<-4$, $\dfrac{|x+4|}{x+4} = \dfrac{-(x+4)}{x+4} = -1 \Rightarrow \lim_{x\to -4^-}\dfrac{|x+4|}{x+4} = -1$

& for $x>-4$, $\dfrac{|x+4|}{x+4} = \dfrac{x+4}{x+4} = 1 \Rightarrow \lim_{x\to -4^+}\dfrac{|x+4|}{x+4} = 1$

$\Rightarrow \lim_{x\to -4}\dfrac{|x+4|}{x+4}$ $\boxed{\text{does not exist}}$

f) $\lim_{x\to 3/2}\dfrac{(2x^2-3x)^2}{|2x-3|} = \lim_{x\to 3/2}\dfrac{x^2(2x-3)^2}{|2x-3|} = \lim_{x\to 3/2} x^2|2x-3| = \boxed{0}$

g) for $x<0$, $\dfrac{2x-1}{\sqrt{3x^2+x+1}} = \dfrac{2-1/x}{\sqrt{3+\frac1x+\frac1{x^2}}}$ and so

$\lim_{x\to -\infty}\dfrac{2x-1}{\sqrt{3x^2+x+1}} = \boxed{2/\sqrt{3}}\left(=\dfrac{2\sqrt3}{3}\right)$

2) $f(x) = \begin{cases} x^2 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$.

Note that $0 \le f(x) \le x^2$ for all x, and since $\lim_{x\to 0} 0 = \lim_{x\to 0} x^2 = 0$, by the "squeeze theorem"

$\lim_{x\to 0} f(x) = 0$.

3) $\lim_{t\to -1^-} \frac{|2t^3-t|}{t^3-t} = \lim_{t\to -1^-} \frac{t-2t^3}{(t^2-1)t} = -\infty$

$\lim_{t\to -1^+} \frac{|2t^3-t|}{t^3-t} = \lim_{t\to -1^+} \frac{t-2t^3}{(t^2-1)t} = +\infty$

vertical asymptote at $t=1$

$\lim_{t\to 0^-} \frac{|2t^3-t|}{t^3-t} = \lim_{t\to 0^-} \frac{|t(2t^2-1)|}{t(t^2-1)} = \lim_{t\to 0^-} \frac{t(2t^2-1)}{t(t^2-1)} = 1$

$\lim_{t\to 0^+} \frac{|2t^3-t|}{t^3-t} = \lim_{t\to 0^+} \frac{-t(2t^2-1)}{t(t^2-1)} = -1$

$\lim_{t\to 1^-} \frac{|2t^3-t|}{t^3-t} = \lim_{t\to 1^-} \frac{2t^3-t}{(t^2-1)t} = -\infty$

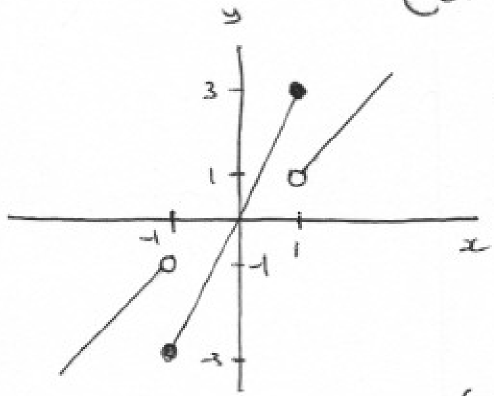
$\lim_{t\to 1^+} \frac{|2t^3-t|}{t^3-t} = \lim_{t\to 1^+} \frac{2t^3-t}{(t^2-1)t} = \infty$

vertical asymptote at $t=1$

$\lim_{t\to \infty} \frac{|2t^3-t|}{t^3-t} = \lim_{t\to \infty} \frac{2t^3-t}{t^3-t} = \lim_{t\to \infty} \frac{2-1/t^2}{1-1/t^2} = 2$ horizontal asymptote $y=2$ ($x\to\infty$)

$\lim_{t\to -\infty} \frac{|2t^3-t|}{t^3-t} = \lim_{t\to -\infty} \frac{t-2t^3}{t^3-t} = \lim_{t\to -\infty} \frac{1/t^2-2}{1-1/t^2} = -2$ horizontal asymptote $y=-2$ ($x\to-$

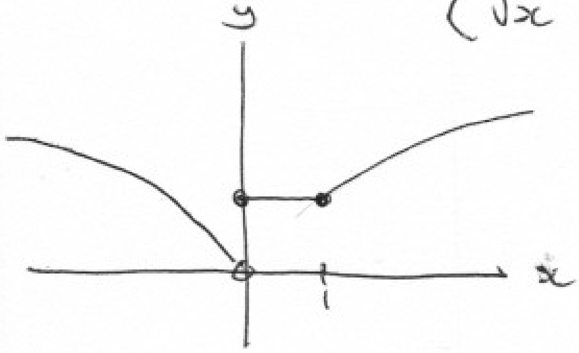
4) a) $f(x) = \begin{cases} 2x+1 & x<-1 \\ 3x & -1 \le x \le 1 \\ 2x-1 & x>1 \end{cases}$



• f is discontinuous at $x=-1$ (right cont. but not left cont.)

and $x=1$ (left cont. but not right cont.)

b) $f(x) = \begin{cases} \sqrt{-x} & x<0 \\ 1 & 0 \le x \le 1 \\ \sqrt{x} & x>1 \end{cases}$



• f is discontinuous at $x=0$ (right. cont. but not left cont.)

5) $g(x)-1 = -x^5-x^4+x^2 = x^2(\underbrace{-x^3-x^2+1}_{f(x)})$

so $g(0)=1$. Note $f(0)=1$ and $f(1)=-1$. Since f is a polynomial, it is continuous, and so by the intermediate value theorem there is some $c \in (0,1)$ such that $f(c)=0$, and hence $g(c)=1$.

5) a) Let $\varepsilon > 0$.

$$|5+3x-(-4)| = |9+3x| = 3|x+3|.$$

So if $0 < |x+3| < \delta := \varepsilon/3$, then

$$|5+3x-(-4)| < 3 \cdot \varepsilon/3 = \varepsilon. \text{ Thus } \lim_{x \to -3} (5+3x) = -4.$$

b) Let $\varepsilon > 0$. $|x^3 - 8| = |(x-2)(x^2+2x+4)|$.

If $|x-2| < \delta$ and $\delta \leq 1$, then $-1 < x-2 < 1 \Rightarrow 1 < x < 3$

so $|x^2+2x+4| \leq |x|^2 + 2|x| + 4 < 9 + 6 + 4 = 19$.

If we set $\delta = \min(1, \varepsilon/19)$, then $|x^3 - 8| < \frac{\varepsilon}{19} \cdot 19 = \varepsilon$.

Hence $\lim_{x \to 2} x^3 = 8$.

c) Let $\varepsilon > 0$. $\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{M\,|g(x)|}$.

Since $\lim_{x \to a} g(x) = M$, $\exists\, \delta > 0$ s.t. if $0 < |x-a| < \delta$, then

$$|g(x) - M| < \min\left(M/2,\ M^2\varepsilon/2\right)$$

and hence $|g(x)| = |M + g(x) - M| \geq M - |g(x) - M|$
$> M - M/2 = M/2$

and so

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| < \frac{M^2\varepsilon/2}{M \cdot M/2} = \varepsilon. \quad \text{Hence } \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}.$$