Chapter: Eigenvalues & Eigenvectors

Def: Let $A$ be an $n \times n$ (square) matrix, a scalar $\lambda \in \mathbb{R}$ or $\mathbb{C}$, and a non-zero vector $\mathbf{v}$ (in $\mathbb{R}^n$ or $\mathbb{C}^n$), are an eigenvalue/eigenvector pair, if

$$A\mathbf{v} = \lambda \mathbf{v}$$

Remarks:

\[ \text{(1)}: \mathbf{v}^T \neq \mathbf{0}^T, \text{ but } \mathbf{v} \text{ can be } \mathbf{0}, \forall \mathbf{v} \in \mathcal{R} (A). \]

and then the corresponding eigenvector for

\[ \text{(2)}: A \mathbf{v} = \lambda \mathbf{v} \Rightarrow (A - \lambda I) \mathbf{v} = \mathbf{0} \]

$d$ $\Rightarrow \mathbf{v} \in \mathcal{R} (A - \lambda I)$.

since $\mathbf{v} \neq \mathbf{0}$, this implies

$\det (A - \lambda I) = 0$
To find the eigenvalues of $A$, find all solutions of

$$P(\lambda) = \det(A - \lambda I) = 0.$$ 

det is a polynomial of degree $n$, called the characteristic polynomial.

That is, set $P(\lambda) = 0$ to find the roots (in complex numbers, possibly repeated).

Denote these roots by $\lambda_1, \lambda_2, \ldots, \lambda_j$

eigenvalues of $A$.

(b) For each $\lambda_j$, solve $(A - \lambda_j I)v = 0$,
i.e. finding $\ker(A - \lambda_j I)$. This subspace is called the eigenspace $E_{\lambda_j}$.

Any basis for $E_{\lambda_j}$ will give us an eigenvector corresponding to $\lambda_j$. 

Ex: Find the eigenvalues/eigenvectors of
\[ A = \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & 2 & 1 \end{bmatrix}. \]

\[ \det(A - \lambda I) = \pm (\lambda - 2)(\lambda - 4)(\lambda - 6) \]

Look at corresponding eigenvectors for \( \lambda_1 = 2, \lambda_2 = 4 \) and \( \lambda_3 = 6 \).

\[ \lambda_1 = 2: \quad E_{\lambda_1} = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\} \]

Here, we have a 3x3 matrix, with 3 eigenvalues. It turns out that \( \lambda_1, \lambda_2, \lambda_3 \) are lin. indep., thus, they...
form a basis of \( \mathbb{R}^3 \) (eigenbasis).

**Theorem:** Let \( A \) be \( n \times n \). If all eigenvalues of \( A \) are distinct, then the corresponding eigenvectors form a basis for \( \mathbb{C}^n \), given by \( \{v_1, \ldots, v_n\} \).

Other possibilities: repeated eigenvalues.

A s.t. \( p(x) = (x-2)^2 (x-3) \).

Complex eigenvalues are not a problem if \( A \) is real, complex eigenvalues come in complex-conjugate pairs, i.e. \( \lambda_1 = a + bi; \quad \bar{\lambda}_1 = a - bi \) and \( \lambda_2 = a - bi; \quad \bar{\lambda}_2 = a + bi \). How about repeated?
We define the algebraic multiplicity of eigenvalues:
\[ m_1 = 2, \quad m_2 = 1, \quad m_3 = 3 \]

Next, \( \text{dim}(E_{d_j}) = d_j \) and the geometric multiplicity of \( d_j \).

Fact: \( 1 \leq d_j \leq m_j \)

Ex: \( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \).