Def. Let \( S_1 \) and \( S_2 \) be two subspaces. Then \( S_1 \) and \( S_2 \) are orthogonal iff
\[ \langle \mu, \nu \rangle = 0. \]
In that case, we write \( S_1 \perp S_2 \).

Remark: Two subspaces are orthogonal iff their basis vectors are orthogonal, i.e., if \( B = \{ b_1, \ldots, b_k \} \) is a basis for \( S_1 \).
Let $\mu \in S_1$, $\nu \in S_2$ that are non-zero.

$\langle \mu, \nu \rangle = 0 \iff \mu \perp \nu$

$\Rightarrow S_1 \perp S_2$ be written down.

**Def:** Let $S_1$ and $S_2$ be two subspaces. Then $S_1$ and $S_2$ are orthogonal iff

Remark: Two subspaces are orthogonal iff their basis vectors are orthogonal, i.e. if:

$B = \{b_1, \ldots, b_k\}$ is a basis for $S_1$
$C = \{c_1, \ldots, c_e\}$ is a basis for $S_2$, then $S_1 \perp S_2 \iff \langle b_i, c_j \rangle = 0$ for $i = 1 \ldots k$, $j = 1 \ldots e$.

Note: This condition can be expressed using matrix notation: set

$B = [b_1 \mid \ldots \mid b_k] \in \mathbb{R}^{m \times k}$

$b_j \in \mathbb{R}^m$ (for $k \leq m$)

$C = [c_1 \mid \ldots \mid c_e] \in \mathbb{R}^{m \times l}$

$(e \leq m)$.

$B^T = \begin{bmatrix}
  b_1^T \\
  \vdots \\
  b_k^T
\end{bmatrix}$

then

$B^T C = \begin{bmatrix}
  b_1^T c_1 & b_2^T c_1 & \cdots & b_e^T c_1 \\
  b_1^T c_2 & b_2^T c_2 & \cdots & b_e^T c_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  b_1^T c_l & b_2^T c_l & \cdots & b_e^T c_l
\end{bmatrix}$

Remember: $b_i^T c_j = \langle b_i, c_j \rangle$

so $S_1 \perp S_2 \iff \text{zero matrix}$
Def: Let $U$ be a subspace of a vector space $\mathcal{W}$. We define

$$U^\perp := \{ w \in \mathcal{W} : \langle w, u \rangle = 0 \}$$

For $U = 0 \times \mathbb{R}^\infty, \langle w, u \rangle = 0$)

$U^\perp$ is called the orthogonal complement of $U$ in $\mathcal{W}$. Note (show yourself) that $U^\perp$ is a subspace.

Remarks:
1. Given $U$ and $U^\perp$ in $\mathcal{W}$, any $x \in \mathcal{W}$ can be written as $x = x_U + x_{U^\perp}$ (uniquely).

In the example above,

$S_1 = S_2^\perp$

Ex: $U = \text{span}\{e_1, e_3\}$ in $\mathcal{W} = \mathbb{R}^5$ ($e_i = (0, \ldots, \underbrace{1}_{i\text{th position}}, \ldots, 0)$).

$U^\perp = \text{span}\{e_2, e_4, e_5\}$
2. \((u^+)^+ = u\)

3. If \(U \subseteq W, \text{dim} W = n\) then \(\text{dim}(u^+) = n - \text{dim}(U)\)

Relation to \(\text{cN}(A), \text{N}(A^T)\)

\(\text{R}(A), \text{R}(A^T)\)

\((1) \text{N}(A) = [\text{R}(A^T)]^+ \perp\)

\((2) \text{N}(A^T) = [\text{R}(A)]^+ \perp\)

Proof of (1):

\((a) [\text{R}(A^T)]^+ \subseteq \text{N}(A) \perp\)

\((b) \text{N}(A) \subseteq [\text{R}(A^T)]^+\)

Let's show (a):

Let \(y \in [\text{R}(A^T)]^+\) be arbitrary.

\(y \perp \text{R}(A^T)\)

\(\mu \in \text{R}(A^T) \Rightarrow \exists x, x \in \mathbb{R}^m, \mu = A^T x\)

\(y \perp \text{R}(A^T)\)

\(0 = y^T x \in \mathbb{R}^m, \langle y, A^T x \rangle = 0\)
\( \forall x \in \mathbb{R}^m, \langle Ay, x \rangle = 0 \)

Let \( y \in \mathbb{R}^m \), not true for \( Ay \)

\[ \Rightarrow \langle Ay, Ay \rangle = 0 \]

\[ \Rightarrow \|Ay\|_2 = 0 \]

\[ \Rightarrow Ay = 0 \]

\( y \in \text{CN}(A) \).

\[ \Rightarrow [R(AT)]^{-1} \subseteq \text{CN}(A) \] \( \subseteq \)

Next:

Let \( x \in \text{CN}(A) \) arbitrary.

Then \( Ax = 0 \)

\[ \Rightarrow \forall y \in \mathbb{R}^m, \langle y, Ax \rangle = 0 \]

\[ \Rightarrow \forall y \in \mathbb{R}^m, \langle A^Ty, x \rangle = 0 \]

\[ \in R(AT) \]

\[ \Rightarrow x \perp R(AT) \]

\[ \Rightarrow x \in [R(AT)]^\perp \]

\[ \Rightarrow [CN(A)]^\perp \subseteq [R(AT)]^\perp \]
Finally, 
\[ R(A^T)^\perp \subseteq \text{null}(A) \]
and 
\[ \text{null}(A) \subseteq [R(A^T)]^\perp \]
so \[
\text{null}(A) = [R(A^T)]^\perp
\]

An immediate use:

\[ Q: \text{let } A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & 5 & 1 \\ \end{pmatrix} \]
Does there exist \( x \) such that 
\[ Ax = \begin{pmatrix} 3 \\ 1 \\ \end{pmatrix} \]?

Typical way of solving:

\[ [A : \begin{pmatrix} 2 \\ 3 \end{pmatrix}] \text{ no RREF (Gaussian)} \]

Repeat the question:
Is \((\frac{2}{3})\) in \( R(A) \)?

Another equivalent question:
Is \((\frac{2}{3})\) in \([\text{null}(A^T)]^\perp \) in \( R(A) \)?