3. Consider the plane $S$ defined by $2u - 3v + w = 0$, and recall that the normal to this plane is the vector $a = [2, -3, 1]$.

(a) Compute the projections of vectors $[1, 0, 0]$ and $[0, 1, 0]$ onto the line spanned by $a$.

**Solution:** The projection matrix is $P = \frac{1}{\|a\|^2}aa^T = \frac{1}{14} \begin{bmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{bmatrix}$ so the projections are

$$p_1 = P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix}$$ and
$$p_2 = P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -6 \\ 9 \\ -3 \end{bmatrix}.$$

(b) Compute the projections of vectors $[1, 0, 0]$ and $[0, 1, 0]$ onto the subspace defined by $S$. What is the inner product of each of these projections with $[2, -3, 1]$?

**Solution:** The complementary projection is $Q = I - P$ so the projections are $q_1 = (I - P) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{14} \begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 10 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -6 \\ 9 \\ -3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 6 \\ 5 \end{bmatrix} \end{bmatrix}$.

The inner product of each of these projections with $[2, -3, 1]$ is zero.
(c) Find a basis for the subspace of $\mathbb{R}^3$ defined by $S$. What is the dimension of this subspace?

**Solution:** The vectors $q_1$ and $q_2$ form a basis. The dimension of this subspace is 2.

(d) The *reflection* of vector $x$ across a subspace is $(2P - I)x$ where $I$ is the identity matrix and $P$ is the matrix projecting $x$ onto the subspace.

i. Draw a sketch to show why this definition of reflection makes sense.

ii. What is the reflection of $[1, 0, 0]$ in plane $S$?

iii. What is the matrix $(2P - I)^2$?

**Solution:**

i. [sketch]

ii. The matrix $P$ in this question is the $Q$ of part (b). $(2Q - I) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2q_1 - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -4 \end{bmatrix}$

iii. $(2P - I)^2 = 4P^2 - 4P + I = 4P - 4P + I = I$ This makes sense because reflecting twice results in the original vector.
4. Consider the following bivariate data:

\[
\begin{array}{cc}
\text{x} & \text{y} \\
-1 & 0 \\
1 & 1 \\
3 & 1 \\
\end{array}
\]

(a) Draw a sketch showing the approximate least-squares straight-line fit \( y = ax + b \) to this data.

**Solution:** [sketch]

(b) Write down the least squares (or normal) equation satisfied by \([a \ b]\).

**Solution:** The equation is \( A^T A [a \ b] = A^T y \) where \( A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \) and \( y = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \). Explicitly \[
\begin{bmatrix} 11 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.
\]

(c) What quantity is minimized by the solution to the equation in (b)?

**Solution:** The minimized quantity is \( \| A \begin{bmatrix} a \\ b \end{bmatrix} - y \|^2 = (-a + b - 0)^2 + (a + b - 1)^2 + (3a + b - 1)^2 \).
[13] 4. Let $S$ be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \\ -1 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. Given the MATLAB/Octave calculation:

```matlab
> rref([1 2 4; 1 -1 1; 1 -1 1])
ans =
    1   0   2
    0   1   1
    0   0   0
```

(a) [7 pts] Find the matrix $P$ that projects onto $S$.

(b) [6 pts] Write down the MATLAB/Octave commands that find the vector in $S$ closest to $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. 


3. Let \( S = \{ [x_1, x_2, x_3]^T : x_1 + x_2 + x_3 = 0 \} \) be the subspace of vectors in \( \mathbb{R}^3 \) whose components sum to zero.

(a) [2 pts]
Find a matrix \( A \) so that \( S \) is the null space of \( A \), i.e., \( S = \text{N}(A) \).

(b) [3 pts]
Write down a basis for \( S \).

(c) [3 pts]
Find a matrix \( B \) so that \( \text{R}(B) = S \)
(d) [4 pts]
Write down the MATLAB/Octave code that
(i) computes the projection matrix $P$ that projects onto $S$ and
(ii) computes the vector in $S$ that is closest to $[0,1,0]^T$.

(e) [4 pts]
Let $Q = I - P$. What kind of matrix is $Q$? What are $N(Q)$ and $R(Q)$?
Exercises Projection

Problem 1. Let $P$ be the orthogonal projection matrix that projects vectors onto the subspace $S$. In terms of $P$, what is the matrix $R$ that performs the orthogonal reflection across the subspace $S$? For example, if $S$ is the $x_1$-$x_2$ plane, $R$ would map the vector $\mathbf{x} = (1, 1, 1)^T$ to the vector $(1, 1, -1)^T$ (i.e., $\mathbf{x}$ is reflected across the $x_1$-$x_2$ plane). Do not derive $R$ for this special case - obtain an expression for $R$ in terms of $P$ for a general case.
**Problem 2.** Here, we will show that $A^T A$ is invertible if and only if $A x = 0$ has only the trivial solution. $A$ need not be square.

1. Show that if there exists a vector $x$ such that $A x = 0$, then $A^T A x = 0$.

2. Show that if there exists a vector $y$ such that $A^T A y = 0$, then we must have $A y = 0$.

3. Explain why we may conclude from (a) and (b) that $N(A) = N(A^T A)$.

4. Use (c) to conclude that $A^T A$ is invertible if and only if $A x = 0$ has only the trivial solution.
Problem 3. From class, we claimed that the least squares solution $\bar{x}$ to $A^T A \bar{x} = A^T b$ always exists, even when $A^T A$ is not invertible. That is, the column space of $A^T A$ is the same as the column space of $A^T$. Show that this is true. Hint: use the result from (3).
Problem 4. Suppose that the columns of an $m \times n$ matrix $A$ form a basis for a certain subspace $S$, and that the columns of the $m \times n$ matrix $B$ also form a (different) basis for the same subspace. Show that $A(A^T A)^{-1} A^T = B(B^T B)^{-1} B^T$. That is, show that the matrix that projects onto $S$ is unique, independent of the choice of basis. Hint: think about how the matrices $A$ and $B$ must be related. You can find hints from Ch. 2 lecture notes.
Problem 5. Let $A$ and its row-reduced echelon form be given by

$$A = \begin{pmatrix} 1 & 3 & 2 & -8 \\ 5 & 15 & 6 & -32 \\ -1 & -3 & 2 & 0 \\ 3 & 9 & 2 & -16 \end{pmatrix}, \quad A \sim \begin{pmatrix} 1 & 3 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Find the projection matrix $P$ that projects vectors onto the column space of $A$. You may use Matlab to perform all your computations. Explain all your steps and give the Matlab commands that you type.
Problem 6. \( P \mathbf{x} \) is the closest vector in \( R(P) \) to the vector \( \mathbf{x} \). Here, \( P \) denotes the projection matrix that projects vectors onto \( R(P) \). We did so by showing that the square of the distance from \( P \mathbf{y} \) to \( \mathbf{x} \) is given by

\[
d^2 = \|P(\mathbf{y} - \mathbf{x})\|^2 + \|Q\mathbf{x}\|^2,
\]

where \( Q = I - P \) is the projection matrix that projects vectors onto \( R(P) \). We then claimed that \( \|P(\mathbf{y} - \mathbf{x})\|^2 \) is minimized when \( P(\mathbf{y} - \mathbf{x}) = \mathbf{0} \), and this occurs when \( \mathbf{y} = \mathbf{x} \). However, \( P \) is in general not invertible, and so has a nontrivial null space.

1. In terms \( Q \), write the set of all possible solutions \( \mathbf{y} \) to the system \( P(\mathbf{y} - \mathbf{x}) = \mathbf{0} \).

2. Is \( P \mathbf{x} \) still the only closest point in \( R(P) \) to \( \mathbf{x} \), or are there other points in \( R(P) \) that are equally close to \( \mathbf{x} \)? Explain your answer.
Problem 7. For a healthy child, the systolic blood pressure $p$ (in millimeters of mercury) and weight (in pounds) are approximately related by the simple model

$$p = \beta_0 + \beta_1 \log w,$$

where $\log w$ denotes the natural logarithm of $w$. Here, $\beta_0$ and $\beta_1$ are constant parameters of the model to be found. You are given sample measurement data for $w$ and $p$ (on the course website). Save the data as a .mat file. Refer to the code from the Mar. 09 lecture for how to load the data in Matlab.

1. Use the least squares data fitting procedure to find values $\beta_0$ and $\beta_1$ that best fit the data. You do not need to turn in any Matlab code or any plots. However, to verify that you have the correct $\beta_0$ and $\beta_1$ values, it would be wise to plot the model from Eqn. (1) against the data.

2. Use the results of (a) to estimate the systolic blood pressure for a healthy child weighing 85 lbs.