2. (10 pts) Consider the differential equation

\[ f''(x) + xf(x) = 1, \quad f(2) = 0, \quad f'(8) = 5 \]

in the interval \( 2 \leq x \leq 8 \).

(a) Let \( F = [F_0, F_1, F_2, F_3] \) be the finite differences approximation to \( f(x) \) when \( N = 3 \). Write down the matrix equation that \( F \) must solve. (Note that the second boundary condition involves the derivative, \( f'(8) \).)
(b) (10 pts) Now suppose \( N = 1000 \). Below is partial Matlab code to construct the finite differences matrix equation and solve it. Some pieces of code are missing. Fill in the missing pieces. In particular, any time you see blank lines of the form _______________, there is a piece of code for you to fill in. (You don’t necessarily have to use all of the lines.) Also, there is one error somewhere in the code that is already written. Find it and fix it.

```matlab
>> N = 1000;
>>
>> % This program sets up and solves a matrix equation of the form
>> % A F = b, where F is a vector of length N+1 representing the
>> % discrete approximation to f.
>>
>> % We first set deltaX to be the length of the interval between
>> % points in our discretization
>>
>> deltaX = _____________
>>
>> % We now construct the matrix L. We first set it to all zeros
>> % and then add in entries.
>> L = zeros(N+1, N+1);
>>
>> % Now we construct Q
```

```matlab`
%>> D = diag([0, -2*ones(1, N - 1), 0]);
%>> UD = diag([0, ones(1, N)], 1);
%>> LD = diag([ones(1,N), 0], -1);
%>> L = D + UD + LD;
%>
%>> % Now we construct Q
```
>> % Now we construct the matrix A.
>>
>> A = ________________________________
>>
>> % Now we construct the vector b
>>
>> ________________________________
>>
>> ________________________________
>>
>> ________________________________
>>
>> % Now we solve the finite differences matrix equation for F
>>
>> F = _____________
Problem 3. [12] Given three data points \((x_0, y_0) = (0, 0), (x_1, y_1) = (1, 3), (x_2, y_2) = (3, 1)\), consider a function \(f(x)\) that interpolates the data points and is of the form
\[
\begin{align*}
    f(x) &= \begin{cases} 
    p_1(x) = a_1 x^3 + b_1 x^2 + c_1 x + d_1, & 0 \leq x \leq 1 \\
    p_2(x) = a_2 x^3 + b_2 x^2 + c_2 x + d_2, & 1 \leq x \leq 3.
    \end{cases}
\end{align*}
\]

(a) Write down the equations that \(a_j, b_j, c_j\) and \(d_j\) must satisfy so that \(f(x)\) passes through the given data points and is continuous.

(b) Let the vector \(a\) be defined as \(a := [a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2]^T\) and express the above system of equations as a matrix equation.
(c) Now suppose that the functions $f'$, $f''$ are continuous at 1; $f''(1) = 1$; and $f''(3) = 2$. Write down the equations that $a_j$, $b_j$, $c_j$ and $d_j$ must satisfy for these conditions to hold. **You do NOT need to rewrite the equations that already appeared in part (a).**

(d) Express the full system of equations you found in parts (a)-(c) as a matrix equation. How can you check in MATLAB if this system has a unique solution?
(e) (3 points) Suppose $C$ is a $3 \times 3$ matrix with $\text{cond}(C) = 10$. If $C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $C \begin{bmatrix} 1 \\ 1+a \\ 1 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 0 \\ 0 \end{bmatrix}$, what are the possible values of $a$?

2. Let $(x_i, y_i), i = 1, \ldots, 4$ be four points in the plane with $x_1 < x_2 < x_3 < x_4$.

(a) (3 points) If the polynomial $p(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4$ interpolates the four points, then the coefficient vector $\mathbf{a} = [a_1, a_2, a_3, a_4]^T$ satisfies an equation of the form $A\mathbf{a} = \mathbf{d}$. Write down $A$ and $\mathbf{d}$. 
(b) (3 points) If the polynomial \( p(x) = b_1 x^4 + b_2 x^3 + b_3 x^2 + b_4 x + b_5 \) interpolates the four points, and also satisfies \( p'(x_4) = 0 \), then the coefficient vector \( \mathbf{b} = [b_1, b_2, b_3, b_4, b_5]^T \) satisfies an equation of the form \( B \mathbf{b} = \mathbf{d} \). Write down \( B \) and \( \mathbf{d} \).

(c) (4 points) If the polynomial \( p(x) = c_1 x^2 + c_2 x + c_3 \) interpolates the four points, then the coefficient vector \( \mathbf{c} = [c_1, c_2, c_3]^T \) satisfies an equation of the form \( C \mathbf{a} = \mathbf{e} \). Write down \( C \) and \( \mathbf{e} \).
(d) (4 points) For each of the equations in parts (a) (b) and (c) say whether you expect there to be a solution. For the case(s) where you do not expect a solution, write down the least squares equation. Do these have a solution? Give a reason. What quantity is minimized when the least squares equation is satisfied?
2. We wish to interpolate the points \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\) with \(x_1 < x_2 < x_3\) using a function of the form
\[
f(x) = \begin{cases} 
  a_1 x^2 + b_1 x + c_1 & \text{for } x_1 < x < x_2 \\
  a_2 x^2 + b_2 x + c_2 & \text{for } x_2 < x < x_3 
\end{cases}
\]

(a) Write down the equations satisfied by \(a_1, b_1, c_1, a_2, b_2, c_2\) when \(f(x)\) is continuous and passes through the given points.

Solution:
\[
\begin{align*}
x_1^2 a_1 + x_1 b_1 + c_1 &= y_1 \\
x_2^2 a_1 + x_2 b_1 + c_1 &= y_2 \\
x_2^2 a_2 + x_2 b_2 + c_2 &= y_2 \\
x_3^2 a_2 + x_3 b_2 + c_2 &= y_3 
\end{align*}
\]

(b) Write down the equation satisfied by \(a_1, b_1, c_1, a_2, b_2, c_2\) when \(f'(x)\) is continuous at \(x = x_2\).

Solution:
\[
2 x_2 a_1 + b_1 - 2 x_2 a_2 - b_2 = 0
\]

(c) Write down the matrix \(A\) and the vector \(b\) in the matrix equation \(Aa = b\) satisfied by \(a = [a_1, b_1, c_1, a_2, b_2, c_2]^T\) when the conditions of both (a) and (b) are satisfied and when \(x_1 = 0, x_2 = 1, x_3 = 2, y_1 = 1, y_2 = 3, y_3 = 2\). Explain why this system of equations does not have a unique solution.

Solution: \(A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 1 \\ 2 & 1 & 0 & -2 & -1 & 0 \end{bmatrix}\) and \(b = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}\). Since \(A\) is a \(5 \times 6\) matrix its null space must be at least one dimensional. This implies any solution will not be unique.
(d) Let \( A \) and \( b \) be as in (c) and assume they have been defined in MATLAB/Octave. Using that \( a=A\backslash b \) computes a solution (even if it is not unique) and \( n=\text{null}(A) \) computes a vector in \( N(A) \), write the MATLAB/Octave code that computes and plots two different interpolating functions of the form \( f(x) \) satisfying the conditions in (a) and (b).

Solution:

\[
\begin{align*}
a &= A \backslash b; \\
n &= \text{null}(A); \\
X_1 &= \text{linspace}(0,1,100); \\
\text{plot}(X_1,\text{polyval}(a(1:3),X_1)) \\
\text{hold on} \\
X_2 &= \text{linspace}(1,2,100); \\
\text{plot}(X_2,\text{polyval}(a(4:6),X_2)) \\
a_1 &= a + n \\
X_1 &= \text{linspace}(0,1,100); \\
\text{plot}(X_1,\text{polyval}(a_1(1:3),X_1)) \\
\text{hold on} \\
X_2 &= \text{linspace}(1,2,100); \\
\text{plot}(X_2,\text{polyval}(a_1(4:6),X_2)) \\
\text{hold off}
\end{align*}
\]
We want to interpolate through \((0, 1), (1, 0), (2, 2)\) using cubic splines

\[
f(x) = \begin{cases} 
p_1(x) & 0 \leq x \leq 1 \\
p_2(x) & 1 \leq x \leq 2 
\end{cases}
\]

(a) [4 pts] Write down \(p_1(x)\) and \(p_2(x)\) in terms of unknown coefficients.

(b) [4 pts] \(f(x)\) must pass through all given points. Write down the linear equations expressing this condition.

(c) [4 pts] Interior derivative must be continuous. Write down the linear equations expressing this condition.

(d) [4 pts] At the endpoints we have zero second derivatives. Write down the linear equations expressing this condition.

(e) [4 pts] Combine equations (a) – (d) into a single matrix equation and write down the MATLAB/Octave commands you need to solve it.
1. In this question we will work with polynomials of degree 3 written
\[ p(x) = a_1x^3 + a_2x^2 + a_3x + a_4 \]

(a) [4 pts]
The coefficient vector \( a = [a_1, a_2, a_3, a_4]^T \) satisfies an equation of the form \( Aa = 0 \) when the slopes of
\( p(x) \) at \( x = 0 \) and \( x = 2 \) are zero. Write down the matrix \( A \).

(b) [4 pts]
Show that \( \dim(N(A)) = 2 \) and find a basis \( a_1, a_2 \) for \( N(A) \).

(c) [4 pts]
The coefficient vector \( a = [a_1, a_2, a_3, a_4]^T \) satisfies an equation of the form \( Ba = b \) when the graph of
\( p(x) \) passes through the points \( (0,1), (1,2) \) and \( (2,2) \). Write down the matrix \( B \) and the vector \( b \).
(d) [4 pts]
Using the equation in (c) find the equation $Cs = c$ satisfied by $s = [s_1, s_2]^T$ if $p(x)$

(i) has coefficient vector $a = s_1a_1 + s_2a_2$ (and therefore has zero slopes at $x = 0$ and $x = 2$).

(ii) passes through the points $(0, 1)$, $(1, 2)$ and $(2, 2)$.

Does this equation have a solution? Give a reason.

(e) [4 pts]
Write down the MATLAB/Octave code that plots the points $(0, 1)$, $(1, 2)$ and $(2, 2)$ and the polynomial $p(x)$ that

(i) has zero slopes at $x = 0$ and $x = 2$.

(ii) comes closest in the least squares sense to passing through the points $(0, 1)$, $(1, 2)$ and $(2, 2)$. 
[18] 2. The boundary value problem

\[ f''(x) + x f(x) = 1, \quad 0 < x < 1 \]
\[ f(0) = 1, \quad f'(1) = 1 \]

can be approximated by an \((N + 1) \times (N + 1)\) system of linear equations of the form

\[ (L + (\Delta x)^2 Q)F = b \]

(a) [10 pts]
Write down \(L, Q, b\) and \(\Delta x\) when \(N = 4\).

(b) [8 pts]
How would you use MATLAB/Octave to compute approximations to \(f(1/2)\) and \(f'(1/2)\)? Assume that \(N\) has been defined and write code that uses this value of \(N\).
Exercises Interpolation

Problem 1.

1. Construct the $9 \times 9$ system of equations for the coefficients of the cubic spline interpolation through the points $(0, 7)$, $(1, 10)$, $(3, 3)$, and $(6, 0)$. Write the system in the form $Ax = b$ and give $A$, $x$, and $b$. As in class, impose the “natural” conditions that the double derivative of the interpolating function must vanish at the end points. Plot the resulting curve, along with the four data points.

2. MATLAB does not impose the two “natural” conditions of the double derivative vanishing at the end points. Instead, the two conditions it imposes are that the third derivative of the interpolating function must be continuous at the second point ($x = 1$) and the second to last point ($x = 3$). Modify your matrix from question 1. to replace the original “natural” conditions with these two new conditions, and give the resulting matrix (only two rows in the matrix need be modified). Plot the curve on the same figure as that from question 1., along with the four data points.
Problem 2. Define the function $G(x; \xi)$ as the solution of the boundary value problem
\[
\frac{d^2 G}{dx^2} = \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{(x-\xi)^2}{2\sigma^2}}, \quad 0 < x < 1; \quad G(0; \xi) = 0; \quad G(1; \xi) = 0. \tag{1}
\]
Here, $G(x, \xi)$ describes the deflection of a string held on both ends at height equal to 0 due to a downward force that is normally (i.e., according to a Gaussian distribution) distributed around the point $\xi$ with variance $\sigma^2$. See figure below.

Figure 1: String of length one (black) held fixed at both ends under load from an external downward force (blue). The length of the arrow is proportional to the magnitude of the force. What is the resulting deflection of the string?

MATLAB can compute integrals using the trapezoidal rule. For example, make a vector $x = \text{linspace}(0,2,1000)$, then define a function, e.g., $y = x.\wedge 2$. Then the value of $\int_0^2 x^2 \, dx$ is approximated by $\text{trapz}(x,y)$. For three values of $\xi \in (0,1)$, use $\text{trapz}()$ to verify that the integral of the function on the right-hand side of Eqn. (1) over $0 \leq x \leq 1$ is approximately 1 when $\sigma = 0.01$.

(a) Use finite differences to solve numerically for $G(x, \xi)$ when $\xi = 0.7$ and $\sigma = 1, 0.1,$ and 0.01. Plot all solutions in one figure. As $\sigma$ becomes small, does the solution fit your intuition?

(b) Use finite differences to solve numerically for $G(x, \xi)$ when $\sigma = 0.01$ and $\xi = 0.2, 0.5$ and 0.7. Plot all solutions in one figure.

(c) Explain why solutions to parts (a) and (b) exist and are unique. You may use MATLAB to compute the determinant of a particular matrix.

(d) For part (b), find the difference in the slope of the string for $x > \xi$ versus $x < \xi$. That is, numerically compute $G_x(\xi + 0.1; \xi) - G_x(\xi - 0.1; \xi)$. This value should be independent of $\xi$.

(e) Compute the deflection of the same string when the loading is given by $f(x) = 1-x^2$. That is, replace the function on the right-hand side of Eqn. (1) by $f(x)$. Plot the solution.
(f) Use \texttt{trapz()} to numerically compute the quantity

\[
u(\xi) = \int_0^1 G(x; \xi)(1 - x^2) \, dx.
\]

Plot \( u(\xi) \) over the interval (0, 1) and compare to the solution in part (??). Submit a printout of your code.
Problem 3. Use finite differences to formulate the boundary value problem

\[ \frac{d^2 u}{dx^2} = b(x), \quad 0 < x < 1; \quad u'(0) = 0; \quad u'(1) = 0, \]  

as a linear system of equations written in the form

\[ Au = b, \]  

and answer the following questions. In Eq.(2), \( u'(a) \) denotes the derivative of \( u \) at \( x = a \). Assume the following: the interval is divided up into \( N + 1 \) evenly distributed points (and therefore \( N \) equally sized intervals) with \( u_0 = u(0) \) and \( u_N = u(1) \).

(a) Give the matrix \( A \) and the vectors \( u \) and \( b \). Be sure to indicate the dimensions of each.

(b) Use MATLAB to compute \( \det A \).

(c) Use MATLAB to determine the dimension of the nullspace of \( A \). Here, I would suggest using a few small values of \( N \) (e.g., 5, 10, 15).

(d) Will a solution to Eqn.(3) exist for very possible vector \( b \)? Explain.

(e) **BONUS:** What condition must \( b \) satisfy in order for there to exist a solution to Eqn.(3)? What is the corresponding condition that \( b(x) \) must satisfy for there to exist a solution to Eqn.(2)?

(f) **BONUS:** When \( b \) satisfies the condition of part (??), the solution to Eqn.(3) (and by extension,(2)), will not be unique. In this case, an additional constraint can be imposed to uniquely specify the solution. Often, this is an integral constraint, e.g.,

\[ \int_0^1 u(x) \, dx = 0. \]  

For Eqn.(2) with the constraint in Eqn. (??), formulate a linear system of equations of the form \( A'u = b' \). Give the matrix \( A' \) and its dimensions. Use this to find the unique numerical solution for \( u \) when

\[ b(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}; \quad \mu = 0.3, \quad \sigma = 0.01. \]
Problem 5. The command \([V, D] = \text{eigs}(A, m, 'lr')\) returns the \(m\) eigenvalues of \(A\) with largest real part along the diagonal of \(D\), and the corresponding eigenvectors in the columns of \(V\). For example, type the following:

\[ A = \text{diag}([1, 4, 6, 8, -10, 5, 1, -15, -3, 12]); \]
\[ [V, D] = \text{eigs}(A, 3, 'lr'); \]
\[ V \]
\[ D \]

It is clear that 12, 8, and 6 are the three largest eigenvalues of \(A\), while the corresponding eigenvectors are \(e_{10}\), \(e_4\), and \(e_3\). You can verify both in \(D\) and \(V\).

(a) Use finite differences with 300 equally spaced points to formulate the following boundary value problem

\[
\frac{d^2 u}{dx^2} + ke^x u = \lambda u \quad 0 < x < 1; \quad u(0) = 0; \quad u(1) = 0,
\]

as an eigenvalue problem of the form

\[
Au = \lambda u.
\]

Give \(A\) and its dimensions.

(b) For \(k = 1\), find the three largest eigenvalues of Eqn. (??), and plot the corresponding eigenvectors (as a function of \(x\)) on one plot.

(c) Repeat for \(k = 0\). Can you guess what the exact answer should be?