Problem Set 9. Due Thursday November 19.

All rings are assumed commutative.

1. Prove that for two ideals $I_1 = \langle d_1 \rangle$ and $I_2 = \langle d_2 \rangle$ in $\mathbb{Z}$, there is an inclusion $I_1 \subseteq I_2$ iff $d_2 | d_1$.

2. Let $I \subset R$ be an ideal in a ring R, and let $\pi : R \to R/I$ be the canonical projection from the ring to the quotient ring (by definition, $\pi(x) = x + I$).

Prove that $J \mapsto \pi(J)$ is a bijection between the ideals J in R containing I, and the ideals in R/I.

3. Prove that a ring R is a field iff R has only two ideals: $\{0\}$ and R.

4. Let R and S be rings with identities 1_R and 1_S, respectively. Let $\phi\colon R \to S$ be a nonzero map satisfying $\phi(x + y) = \phi(x) + \phi(y)$, and $\phi(xy) = \phi(x)\phi(y)$.

(a) Prove that if $\phi(1_R) \neq 1_S$, then $\phi(1_R)$ is a zero-divisor in S.

(b) Find an example of ϕ, R, and S, when $\phi(1_R)$ is a zero-divisor in S.

Note: some sources call such maps ϕ ring homomorphisms. The only difference between this definition and the definition in the text is, the text requires $\phi(1_R) = 1_S$ for the map to be a homomorphism, to exclude these weird examples.

5. Let R and S be rings, and $R \neq \{0\}$. Let $\phi\colon R \to S$ be a ring isomorphism.

(a) Prove that the set of zero divisors in S is equal to the set $\{\phi(r) \mid r \text{ is a zero divisor in } R\}$ (i.e. it is the image of the set of zero divisors in R).

(b) Assume that R and S have identities 1_R and 1_S, respectively. Let S^* be the set of units in S, R^* – the set of units in R. Prove that $S^* = \phi(R^*)$.

6. Let $R = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbb{Z}\}$.
Let $I = \{a + b\sqrt{5} \in R \mid a - b \text{ is divisible by } 4\}$.

(a) Prove that I is an ideal of R.

(b) Prove that the map $\phi\colon R \to \mathbb{Z}/4\mathbb{Z}$ defined by $\phi(a + b\sqrt{5}) = a - b + \langle 4 \rangle$ (where $a - b + \langle 4 \rangle$ is the congruence class of the number $a - b$ in $\mathbb{Z}/4\mathbb{Z}$) is a homomorphism.

(c) Prove that R/I is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

(d) Describe the set of zero divisors in R/I.

(e) Describe the set of units $(R/I)^*$ in R/I.

7. Problem 23 from Section 3.6.