Modular forms, Homework 3. Part 1. Due November 13.

1. Let $E_1(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} + \frac{1}{z-n}\right)$. Let $\gamma_{2k} = 2\sum_{n=1}^{\infty}\frac{1}{n^{2k}}$.

Without using the fact discussed in class that the series is related to the cotangent function, prove that:

(a) Prove that E_1 satisfies the differential equation $E_1'(z) = (-3\gamma_2 + E_1^2)$.

(b) Let $\theta = \sqrt{3\gamma_2}$, and let $u(z) = \frac{1}{\theta}E_1(z/\theta)$. Then $u' = -(1+u^2)$.

(c) Conclude that $E_1(z) = \theta\cot(\theta z)$, and that $\theta = \pi$, yielding the equality $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

2. Show the following identities for the sums of powers of divisors: for each n,

(a)

$$\sigma_7(n) - \sigma_3(n) = 120\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m);$$

(b)

$$11\sigma_9(n) - 21\sigma_5(n) + 10\sigma_3(n) = 5040\sum_{m=1}^{n-1}\sigma_3(n)\sigma_5(n-m).$$

3. Let $\mathbb{G}_k(z)$ be the re-normalized Eisenstein series so that

$$\mathbb{G}_k(z) = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n,$$

where B_{2k} is the Bernoulli number.

(a) Prove that

$$\mathbb{G}_6(z) = \Delta(z) + \frac{691}{156}\left(\frac{E_2(z)^3}{720} + \frac{E_3(z)^3}{1008}\right).$$

(b) Deduce the congruence of Ramanujan:

$$\tau(n) \equiv \sigma_{11}(n) \mod 691.$$

Hint: 691 *comes from the numerator of the constant term* $-B_{12}/24$ *of* $\mathbb{G}_6$.

4. Prove that the cross-ratio used to define the distance on $\mathbb{H}$ (see p. 62 of Milne's notes) is real for any $z_1, z_2 \in \mathbb{H}$.

Hint: You can use the fact that the cross-ratio is invariant under the Möbius transformations.