Modular forms, Homework 2 Part 2. Due October 23.

1. Prove Theorem 3.8 in Milne.

2. Prove that Weierstrass $\wp$-function satisfies the differential equation:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

3. In this problem we establish that an elliptic curve is an *algebraic* group with respect to addition, i.e. that the operation of addition is algebraic. Note that $\mathbb{C}/\Lambda$ is, naturally, a group with respect to addition.

(a) Prove Milne's Proposition 3.9:

$$\wp(z+z') = \frac{1}{4}\left(\frac{\wp'(z)-\wp'(z')}{\wp(z)-\wp(z')}\right)^2 - \wp(z) - \wp(z').$$

(b) Prove that if E is the elliptic curve with $E(\mathbb{C}) = \mathbb{C}/\Lambda$ then the maps $E \times E \to E, (x, y) \mapsto x + y$ and $E \to E, x \mapsto -x$ are algebraic.

4.* **Inversion of an elliptic integral.** As a warm-up, consider the stereographic projection for a circle: you can check (or believe – do not need to write it up) that with a suitable choice of coordinates it is given by $t \mapsto \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right)$ (think of it as a map from the line to the circle). What this formula does is: it provides a *rational* bijection between the set of solutions to the equation $x^2 + y^2 = 1$ and the points of the projective line (note that $\left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right)$ is the affine representation of the point $(t^2 - 1 : 2t : t^2 + 1)$). Note that this bijection works over *any* field of characteristic not 2 (if you do it over $\mathbb{Q}$, you get Pythagorean triples; over a finite field it is useful for counting solutions to $x^2 + y^2 = 1$, and over $\mathbb{C}$ it gives us yet another way to think of the Riemann sphere: namely, as a *curve* in $\mathbb{P}^2(\mathbb{C})$ defined by the equation $x^2 + y^2 = 1$).

(a) Consider the Riemann sphere again, this time, think of it as above – as the curve in $\mathbb{P}^2(\mathbb{C})$ with the equation $z^2 + w^2 = 1$. Let z_0 be some fixed point, and let $F(u) = \int_{z_0}^{u} \frac{1}{w}\, dz$. (Does the integral depend on the path?). Find the inverse function of $F(u)$.
Hint: this question is essentially trivial if you recall trig substitution. It actually explains why the trig substitution works: you parametrize the curve as $z = \sin(u)$, $w = \sin'(u)$. *Note also that* sin *is a function with one period.*

(b) Now consider an elliptic curve $w^2 = z^3 - az - b$. Again consider the integral $F(u) = \int_{z_0}^{u} \frac{1}{w}\, dz$ – the integral is along a path on our curve. Observe that this integral is a multi-valued function (why?). However, it has a well-defined inverse: prove that its inverse is, in fact, a suitable Weierstrass $\wp$-function!