1. Prove Theorem 3.8 in Milne.

2. Prove that Weierstrass $\wp$-function satisfies the differential equation:
   
   \[ \wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \]

3. In this problem we establish that an elliptic curve is an algebraic group with respect to addition, i.e. that the operation of addition is algebraic. Note that $\mathbb{C}/\Lambda$ is, naturally, a group with respect to addition.
   (a) Prove Milne’s Proposition 3.9:
   
   \[ \wp(z + z') = \frac{1}{4} \left( \frac{\wp'(z) - \wp'(z')}{\wp(z) - \wp(z')} \right)^2 - \wp(z) - \wp(z'). \]

   (b) Prove that if $E$ is the elliptic curve with $E(\mathbb{C}) = \mathbb{C}/\Lambda$ then the maps $E \times E \rightarrow E, (x, y) \mapsto x + y$ and $E \rightarrow E, x \mapsto -x$ are algebraic.

4. Inversion of an elliptic integral. As a warm-up, consider the stereographic projection for a circle: you can check (or believe – do not need to write it up) that with a suitable choice of coordinates it is given by $t \mapsto (\frac{t^2 - 1}{t^2 + 1}, \frac{2t}{t^2 + 1})$ (think of it as a map from the line to the circle). What this formula does is: it provides a rational bijection between the set of solutions to the equation $x^2 + y^2 = 1$ and the points of the projective line (note that $(\frac{t^2 - 1}{t^2 + 1}, \frac{2t}{t^2 + 1})$ is the affine representation of the point $(t^2 - 1 : 2t : t^2 + 1)$). Note that this bijection works over any field of characteristic not 2 (if you do it over $\mathbb{Q}$, you get Pythagorean triples; over a finite field it is useful for counting solutions to $x^2 + y^2 = 1$, and over $\mathbb{C}$ it gives us yet another way to think of the Riemann sphere: namely, as a curve in $\mathbb{P}^2(\mathbb{C})$ defined by the equation $x^2 + y^2 = 1$).
   (a) Consider the Riemann sphere again, this time, think of it as above – as the curve in $\mathbb{P}^2(\mathbb{C})$ with the equation $z^2 + w^2 = 1$. Let $z_0$ be some fixed point, and let $F(u) = \int_{z_0}^u \frac{1}{w} \, dz$. (Does the integral depend on the path?). Find the inverse function of $F(u)$.
   
   Hint: this question is essentially trivial if you recall trig substitution. It actually explains why the trig substitution works: you parametrize the curve as $z = \sin(u), w = \sin'(u)$. Note also that $\sin$ is a function with one period.

   (b) Now consider an elliptic curve $w^2 = z^3 - az - b$. Again consider the integral $F(u) = \int_{z_0}^u \frac{1}{w} \, dz$ – the integral is along a path on our curve. Observe that this integral is a multi-valued function (why?). However, it has a well-defined inverse: prove that its inverse is, in fact, a suitable Weierstrass $\wp$-function!