1. *(You do not have to write up this solution if you already know this calculation).*

The goal of this problem is to compute the index \( \Gamma(1) : \Gamma(N) \). For a ring \( R \), we denote by \( \text{GL}_2(R) \) the group

\[
\text{GL}_2(R) := \{ X \in M_2(R) \mid \det(X) \in R^\times \},
\]

where \( M_2(R) \) is the set of \( 2 \times 2 \) matrices with entries in \( R \), and \( R^\times \) is the group of units of \( R \). *Hint: see Example 2.23 in Milne for hints.*

(a) Let \( \mathbb{F}_p \) be the field of \( p \) elements. Prove that \( \# \text{GL}_2(\mathbb{F}_p) = (p^2 - 1)(p^2 - p) \).

(b) Let \( r \in \mathbb{N} \). Prove that \( \# \text{GL}_2(\mathbb{Z}/p^r\mathbb{Z}) = p^{4(r-1)}(p^2 - 1)(p^2 - p) \).

(c) Suppose \( N = \prod_i p_i^{r_i} \) is the prime factorization of \( N \). Prove that \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \simeq \prod_i \text{GL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z}) \).

(d) Find \( \# \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \).

(e) Prove that \( \# \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \varphi(N)^{-1} \# \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \), where \( \varphi \) is Euler’s \( \varphi \)-function.

(f) Find \( \# \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \).

(g) \( \Gamma(1) : \Gamma(N) \), where \( \sim \) denotes the quotient by \( \{ \pm I \} \) if \( -I \) is in the group. Consider the case \( N = 2 \) separately.

2. Exercise 2.24 on p.39 in Milne’s notes.

3. Algebraic description of ramification:

(a) Consider a smooth curve on the affine plane, defined by the equation \( f(x, y) = 0 \), where \( f \) is a degree 2 polynomial. Consider the projection onto the \( x \)-axis. Prove that a point \( (x_0, y_0) \) on the curve is a ramification point for this projection map iff \( \frac{\partial f}{\partial y}(x_0, y_0) = 0 \).

*(Hint: you can use implicit differentiation and consider it a calculus problem.)*

(b) Recall that \( \mathbb{C}P^3 \) is the complex projective space, with homogeneous coordinates \([z_0 : z_1 : z_2 : z_3]\), where \([z_0 : z_1 : z_2 : z_3]\) stands for the equivalence class of triples \((z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \) with the usual equivalence \((z_0, z_1, z_2, z_3) \sim (\lambda z_0, \lambda z_1, \lambda z_2, \lambda z_3) \) with \( \lambda \in \mathbb{C}^\times \) (i.e. the space of lines through the origin in \( \mathbb{C}^4 \)). Let \( X \) be a curve defined by a system of two polynomial equations in \( \mathbb{C}P^3 \), \( p_1(z_0, z_1, z_2, z_3) = p_2(z_0, z_1, z_2, z_3) = 0 \), where \( p_i \) are homogeneous polynomials with complex coefficients. Consider a projective line \( L \) in \( \mathbb{C}P^3 \) given by \( L_1 = L_2 = 0 \) where \( L_1, L_2 \) are linear homogeneous polynomials. Prove that a point on \( X \) is a ramification point for a projection from \( X \) onto \( L \) iff the following Jacobian determinant vanishes at that point:

\[
J := \begin{vmatrix}
\frac{\partial p_1}{\partial z_0} & \frac{\partial p_1}{\partial z_3} \\
\frac{\partial p_2}{\partial z_0} & \frac{\partial p_2}{\partial z_3}
\end{vmatrix} = 0.
\]

3. Using Riemann-Hurwitz formula, prove that the intersection of two generic quadric surfaces in \( \mathbb{C}P^3 \) is an elliptic curve.

More precisely, A *quadric surface* is a surface defined by a degree 2 homogeneous polynomial in these coordinates:

\[
\sum_{0 \leq i, j \leq 3} a_{ij} z_i z_j = 0,
\]
where $a_{ij} \in \mathbb{C}$. By *generic* we mean a property that holds for *almost all* coefficients $(a_{ij})$ (here the notion of ‘almost all’ means, the exceptions form a hypersurface, defined by some polynomial equations, in the space of all coefficients).

For this problem, an *elliptic curve* is a complex projective curve of genus 1. It is OK to work with complex manifolds (and Riemann surfaces) instead of the algebraic surfaces/curves. Thus the problem is asking the following: consider the curve in $\mathbb{CP}^3$ obtained as the intersection of two surfaces defined by equations of the form (1). It is OK to assume without proof that for two generic surfaces, you do get a Riemann surface (i.e. a smooth curve) as the intersection. Then you only need to prove that it has genus 1.

*Hint: use the previous problem. To count ramification points, you can use Bezout’s theorem.*