Modular forms, Homework 1. Due September 25.

**Actions of topological groups**.

*Do not turn in this part unless you really want me to check it. The solutions are in Milne, and in Miyake.*

**1.** Let $G$ be a topological group acting continuously on a topological space $X$.

(a) Show that if $X$ Hausdorff, then for every $x \in X$, the stabilizer of $x$ in $G$ is closed.

(b) Let $H$ be a subgroup of $G$. Prove that the group $G/H$ with the quotient topology is Hausdorff if and only if $H$ is closed in $G$. (Note that it seems that $G$ itself does not need to be assumed Hausdorff for this to hold!).

**2.** Suppose that $G$ acts continuously and *transitively* on $X$. Assume that $G$ and $X$ are locally compact and there is a countable base for the topology on $G$. Prove that:

(a) The map $G/\operatorname{Stab}_G(x) \to X$ is a homeomorphism, for all $x \in X$.

In all of the following, we assume further that for some (hence all) $x \in X$, the stabilizer $\operatorname{Stab}_G(x)$ is compact.

(b) Prove that a subgroup $\Gamma$ of $G$ acts properly discontinuously on $X$ if and only if $\Gamma$ is discrete.

**3.** With all the assumptions of the previous problem, and assuming $\Gamma$ is a discrete subgroup of $G$, show that:

(a) For all $x \in X$, $\operatorname{Stab}_\Gamma(x) := \{\gamma \in \Gamma | \gamma \cdot x = x\}$ is finite.

(b) Every $x \in X$ has a neighbourhood $U$ with the following property: if $\gamma \in \Gamma$ and $U \cap \gamma U \neq \emptyset$, then $\gamma x = x$.

(c) For all $x, y \in X$ that are not in the same $\Gamma$-orbit, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $\gamma U \cap V = \emptyset$ for all $\gamma \in \Gamma$.

(d) $\Gamma \backslash X$ is Hausdorff.

**Riemann sphere and fractional linear transformations.**

**4.** Exercise 2.3 on p.26 of Milne's notes

**5.** Consider the unit sphere $X = \{(a, b, c) : a^2 + b^2 + c^2 = 1\}$ in $\mathbb{R}^3$. Let $N = (0, 0, 1)$, $S = (0, 0, -1)$ be the two poles, and $U_N = X \setminus \{N\}$, $U_S = X \setminus \{S\}$. Consider the following three charts: $\varphi_N : U_N \to \mathbb{C}$, and $\varphi_S, \psi_S : U_S \to \mathbb{C}$ defined by:

$$\varphi_N(a, b, c) = \frac{a + ib}{1 - c}; \quad \varphi_S(a, b, c) = \frac{a + ib}{1 + c}; \quad \psi(a, b, c) = \frac{a - ib}{1 + c}.$$

(a) Calculate $\varphi_N^{-1}(z)$.

(b) Among the three charts $(U_N, \varphi_N)$, $(U_S, \varphi_S)$, $(U_S, \psi_S)$, one pair is compatible and the other is not. Which one is compatible and why? (Hint: *Recall that a real-differentiable function is holomorphic iff it satisfies Cauchy-Riemann equations.*)

**6.** Show that $\mathbb{P}^1(\mathbb{C})$ is isomorphic to the sphere of the previous example (i.e., consider the complex structure on the sphere given by the two compatible charts in the previous problem, and show that it is isomorphic to $\mathbb{P}^1(\mathbb{C})$ as a Riemann surface).

**7.** Let $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be an element of $\mathrm{SL}_2(\mathbb{R})$, let $t \in \mathbb{R}$, and let $U_t := \{z \in \mathbb{H} \mid \Im(z) > t\}$. Describe the image of $U_t$ under $\alpha$.

**8.** Let $\alpha \in \mathrm{SL}_2(\mathbb{R})$.

(a) Prove that the subgroup $\langle\alpha\rangle$ generated by $\alpha$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ if $\alpha$ is parabolic or hyperbolic.
(b) When $\alpha$ is elliptic, $\langle\alpha\rangle$ is discrete if and only if $\alpha$ is of finite order.
(c) Describe the set of elliptic elements in $\mathrm{SL}_2(\mathbb{Z})$.