1. Fourier transform: a reminder

1.1. Functions on the unit circle. For $f \in L^2(S^1)$,

$$\hat{f}(n) = \int_{S^1} f(z) e^{-2\pi i n z}\, dz$$

- converges for $f \in L^2(S^1)$
- The image is a function in $L^2(\mathbb{Z})$.
- It is onto.
- Plancherel formula: $\|f\|^2 = c \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$. ($c$ is a normalization constant; with my choice of measures, $1/2\pi$?).
- The characters of S^1 (i.e., the functions $e^{2\pi i n z}$) form an orthonormal basis in $L^2(S^1)$.

1.2. Functions on $\mathbb{R}$. Define Fourier transform by:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x}\, dx.$$

- When is this guaranteed to converge? When $f \in L^1(\mathbb{R})$ (Riemann-Lebesgue lemma).
- What kind of function of ξ do we get? It is continuous and goes to 0 at ∞, but doesn't have to be in L^1.
- The Schwartz space $\mathcal{S}$, which is contained in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, is taken to itself by Fourier transform.
- What about L^2? The space $\mathcal{S}$ is dense in L^2; it turns out that Fourier transform is continous with respect to the L^2-norm, and therefore, $\mathcal{S}$ being dense in L^2, one can define the Fourer transform for L^2-functions.
- Plancherel formula:
$$\|f\|^2 = c\|\hat{f}\|^2.$$
Note: this is equivalent to Parceval's formula: $\langle f, g\rangle = c\langle \hat{f}, \hat{g}\rangle$.
- Tempered distributions form the dual space to $\mathcal{S}$. Get Fourier transform on tempered distribtutions.
- Why we care for distributions: the characters of $\mathbb{R}$, that is, the functions $e^{2\pi i x}$, would have formed an orthonormal basis of $L^2(\hat{\mathbb{R}})$, if only they were in that space. Instead, they are in the space of tempered distributions.

1.3. Abelian topological groups. We note that if G is finite, the space of functions on G is $\mathbb{C}[G] = C(G) = L^1(G) = L^2(G)$.

The Pontryagin dual of G is the group of unitary characters of G: $\hat{G} = \{\chi : G \to S^1$ – a continuous group homomorphism$\}$; note that that when G is finite, or more generally, compact, the (continuous) characters $\chi : G \to C^\times$ are automatically unitary. Fourier transform is a map from functions on G to functions on $\hat{G}$ defined by:

$$\hat{f}(\chi) = \int_G f(g)\overline{\chi(g)}\, dg,$$

where dg is an Haar measure on G (the integral is just a sum over G when G is finite). Note that this is L^2-inner product of f and χ, so when G is infinite, it makes sense for $f \in L^2(G)$. The two above examples are special cases of this Fourier transform. Indeed, for the example of the functions on the circle, i.e. periodic functions on the interval, we use that $\hat{S^1} \simeq \mathbb{Z}$ (the characters are $z \mapsto z^n$, and we recognize the Fourier coefficients of periodic functions, mentioned above, by identifying the interval $[0, 1)$ with S^1 via $x \mapsto e^{2\pi i x}$). For the functions on the real line, we use the fact that $\hat{R} \simeq \mathbb{R}$ via $y \mapsto \chi_y = (x \mapsto e^{2\pi i x y})$. Note that this isomorphism can be thought of as follows: pick one character of $\mathbb{R}$, say, $\psi(x) = e^{2\pi i x}$. Now, $\chi_y = \psi(xy)$ for $y \in \mathbb{R}$. This identification of the group with its Pontryagin dual works for the additive group of any local field.

1.4. **Mellin transform.** Mellin transform is defined for a function $f : \mathbb{R}_{>0} \to \mathbb{R}$ by:

$$g(s) = \int_0^\infty f(x) x^s \frac{dx}{x}.$$

Note that if we relax the requirement for the character to be unitary, namely, consider the homomorphisms from the multiplicative group of positive reals $\mathbb{R}_{>0}$ to $\mathbb{C}^\times$, then the group of such homomorphisms is $\mathbb{C}$: they are all of the form $x \mapsto x^s$, $s \in \mathbb{C}$. Since dx/x is an invariant measure on $\mathbb{R}_{>0}$ (with respect to the multiplicative group structure), Mellin transform can be thought of as Fourier transform for this group.

1.5. **An exercise on Dirichlet characters: Gauss sums.** Let p be a prime, and let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to S^1$ be a *nontrivial* character of the *multiplicative* group $(\mathbb{Z}/p\mathbb{Z})^\times$. We can extend χ to a function on $\mathbb{Z}/p\mathbb{Z}$ by letting $\chi(0) = 0$. Recall that the group $\mathbb{Z}/p\mathbb{Z}$ is self-dual (i.e, its group of characters is isomorphic to itself via $a \mapsto \chi_a = (x \mapsto \psi(ax))$, where ψ is some chosen non-trivial character). Consider the Fourier transform of the function χ on $\mathbb{Z}/p\mathbb{Z}$ (note that we started with a character of the multiplicative group to get this function, but the Fourier transform is happenning on the additive group $\mathbb{Z}/p\mathbb{Z}$). The Fourier transform $\widehat{\chi} : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ is:

$$\widehat{\chi}(x) = \sum_{y \in \mathbb{Z}/p\mathbb{Z}} \chi(y) e^{-2\pi i x y/p} = \sum_{y \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(y) e^{-2\pi i x y/p}.$$

Let

$$G(\chi) = \sum_{y \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(y) e^{2\pi i y/p}.$$

The sum $G(\chi)$ is called a Gauss sum.

(1) Prove that $\widehat{\chi}(x) = \chi(-1) G(\chi) \overline{\chi}(x)$.
(2) Prove that $\overline{G(\chi)} = \chi(-1) G(\overline{\chi})$.
(3) Prove that $|G(\chi)| = \sqrt{p}$ (note that this implies that there are a lot of cancellations in the sum: a naive estimate of its magnitude would be $|G(\chi)| \leq p - 1$, since it's a sum of $p - 1$ roots of unity).