A supplement on Dirichlet series

Treat this as optional homework – these are all classical facts, just stated as problems. Please think about them; no need to hand in solutions. You can read all the solutions in the "Dirichlet series" chapter of Titchmarsh's textbook (posted).

A *Dirichlet series* is a series of the form $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where a_n and s are complex numbers.

(1) Convergence:
  (a) Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges absolutely for $s = s_0 \in \mathbb{R}$, then it converges absolutely for all $s \in \mathbb{C}$ with $\Re s > s_0$; moreover such convergence is uniform on compact sets contained in the half-plane $\Re s > s_0$. Hence, in the half-plane $\Re s > s_0$ it converges to a holomorphic function.
  (b) Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for $s = s_0 \in \mathbb{R}$, then it converges for all $s \in \mathbb{C}$ with $\Re s > s_0$.
  (c) Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for $s = s_0 \in \mathbb{R}$, then it converges absolutely for all $s \in \mathbb{C}$ with $\Re s > s_0 + 1$.
  (d) Observe that the above statements imply that a Dirichlet series (that converges at least somewhere) has *abscissa of convergence* (i.e. the smallest σ_0 such that it converges for $\Re s > \sigma_0$), and *abscissa of absolute convergence* σ^* (denoted by $\bar{\sigma}$ in Titchmarsh), and that $\sigma^* \leq \sigma_0 + 1$. Now, prove that the abscissa of absolute convergence is given by:

$$\sigma^* = \limsup_{n \to \infty} \frac{\log(|a_1| + \cdots + |a_n|)}{\log n}.$$

(2) (a) Prove that is a function holomorphic in a half-plane $\Re s > 0$ is represented by a Dirichlet series in this half-plane, then this series is unique: if $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ in some half-plane, then $a_n = b_n$ for all n.
  (b) Prove that any function represented by a convergent Dirichlet series has a zero-free half-plane.

(3) Prove that if the coefficients are square summable, i.e., $\sum |a_n|^2 < \infty$, then the series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for $\Re s > 1/2$. (Hint: use Cauchy-Schwarz inequality).

(4) Growth:
  (a) The function $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is bounded in any half-plane strictly contained in the half-plane of absolute convergence.
  (b) If σ_0 is the abscissa of convergence for $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, and σ^* is the abscissa of absolute convergence, then $f(s) = O(|t|^{1-(\sigma-\sigma_0)+\epsilon})$ as $t := \Im(s) \to \infty$, for any value of $\sigma := \Re s$ between σ_0 and σ^*.

(5) Suppose the coefficients are multiplicative: $a_{mn} = a_m a_n$ for all $m, n \geq 1$. Prove that the series then has an Euler product:

$$\sum_{n=1}^{\infty} a_n n^{-s} = \prod_p (1 - a_p p^{-s})^{-1}.$$