Please do not hand in. We can discuss these problems in class if needed.

V will always denote a complex vector space, and $I : V \to V$ – the identity map.
If we say $A : V \to V$, we always mean a linear operator on $V$. Let $V, W$ be two
vector spaces. We denote by $\text{Hom}(V, W)$ the vector space of all linear operators
from $V$ to $W$.

1. JORDAN CANONICAL FORM, AND MISCELLANEOUS PROBLEMS

(1) Let $A : V \to V$ be a linear map. Assume that $A^n = I$ for some $n$. Show
that $V$ has a basis of eigenvectors for $A$ (that is, the matrix of $A$ can be
diagonalized).

(2) * Let $A_s$ be the diagonal part of the canonical Jordan form of $A$, and let
$A_n = A - A_s$. Prove that there exist polynomials $P$ and $Q$, such that
$A_s = P(A)$, $A_n = Q(A)$.

(3) Commuting linear operators.
(a) Suppose $A, B : V \to V$ are diagonalizable linear operators (i.e. each
of them has a basis of eigenvectors). Show that there exists a common
basis of eigenvectors for $A$ and $B$.
(b) Suppose a linear operator $A$ has distinct eigenvalues, and suppose
$AB = BA$. Prove that there exists a polynomial $P$, such that $B =
P(A)$. Is this assertion true if we do not assume that the eigenvalues
of $A$ are distinct?
(c) In general, let $V_\lambda = \cup_m \text{ker}(A - \lambda I)^m$ (call it the generalized eigenspace
of $A$), and suppose $B : V \to V$ commutes with $A$. Show that the
generalized eigenspaces of $A$ are $B$-invariant.

(4) Projectors.
(a) Let $p : V \to V$ be a linear operator satisfying $p^2 = p$ (such operators
are called projectors). Show that there is a direct sum decomposition
$V = \text{ker}(p) \oplus \text{Im}(p)$. (Thus, you can think of $p$ as a projection onto its
image along its kernel).
(b) Let $W$ be a linear subspace of $V$. Show that there is a one-to-one cor-
respondence between projectors $p$ with $\text{Im}(p) = W$, and direct com-
plements of $W$.
(c) Suppose $A : V \to V$ commutes with $p$. Show that $\text{ker}(p)$ and $\text{Im}(p)$
are $A$-invariant subspaces.

2. DUAL VECTOR SPACES AND BILINEAR FORMS

Let $V^*$ denote the linear dual of $V$, i.e., the space of linear functionals
on $V$.

(5) (a) Let $\{e_1, \ldots, e_n\}$ be a basis of $V$. Prove that there exists a basis
$\{e_1^*, \ldots, e_n^*\}$ of $V^*$ with the property $e_i^*(e_j) = \delta_{ij}$. Such a basis is
called the dual basis to $\{e_1, \ldots, e_n\}$. 
(b) Let \( \{e_1, \ldots, e_n\} \) and \( \{e_1^*, \ldots, e_n^*\} \) be dual bases of \( V \) and \( V^* \), respectively. Suppose that \( A : V \to V \) is a linear operator with the matrix \( M = (a_{ij}) \) with respect to the basis \( \{e_1, \ldots, e_n\} \). Let \( A^* : V^* \to V^* \) be the dual linear operator, defined by the property:
\[
A^*(w)(v) = w(Av), \quad \forall w \in W, v \in V.
\]
Show that the matrix of \( A^* \) with respect to the basis \( \{e_1^*, \ldots, e_n^*\} \) is \( (a_{ji}) = M^T \).

(6) Prove that for any matrix \( A \), the rank of \( A \) equals the rank of \( A^T \).

(7) A sequence of linear maps \( V \xrightarrow{A} W \xrightarrow{B} U \) is called exact (in the middle term) if \( \ker(B) = \operatorname{Im}(A) \). A longer sequence is called exact if it is exact in every term.

Prove that the sequence \( 0 \to V \xrightarrow{A} W \xrightarrow{B} U \to 0 \) is exact if and only if the dual sequence \( 0 \to U^* \xrightarrow{B^*} W^* \xrightarrow{A^*} V^* \to 0 \) is exact.

(8) Let \( B : V \times V \to C \) be a linear functional (such linear functionals are called bilinear forms on \( V \)). Find the condition on \( B \) that guarantees that the map \( w \mapsto (v \mapsto B(v, w)) \) is an isomorphism from \( V \) to \( V^* \). (Note that there is no canonical isomorphism from \( V \) to \( V^* \), but any nice enough bilinear form can be used to make such an isomorphism).

(9) Prove that \( \operatorname{Hom}(V, W) \cong \operatorname{Hom}(W^*, V^*) \).

(10) Show that there is a canonical isomorphism \( V^{**} \to V \).

3. Tensor products

(11) Let \( f : V \times W \to V \otimes W \) be the canonical map: \( f(v, w) = v \otimes w \). Prove that it is universal in the following sense:

for any vector space \( U \), and any bilinear map \( B : V \times W \to U \), there exists a unique linear operator \( C : V \otimes W \to U \) such that \( B = C \circ f \).

This is called the universal property of the tensor product. It is not hard to prove that any two objects satisfying such a universal property have to be isomorphic, and thus one can use the universal property as the definition of the tensor product.

(12) Prove that \( V^* \otimes W \) is canonically isomorphic to \( \operatorname{Hom}(V, W) \). (Hint: use the universal property of the tensor product).

(13) (a) Let \( A : V_1 \to V_2 \) be a linear map of vector spaces. Let \( W \) be an arbitrary vector space. Then we can construct the linear map
\[
A \otimes I : V_1 \otimes W \to V_2 \otimes W,
\]
where \( I : W \to W \) is the identity map. Prove that if \( A : V_1 \to V_2 \), \( B : V_2 \to V_3 \) are linear operators, then \((B \circ A) \otimes I = (B \otimes I) \circ (A \otimes I)\).

(Note: this property tells us that “tensoring with \( W \)” is a functor from the category of vector spaces over \( \mathbb{C} \) to itself.)
(b) Suppose \( 0 \to V_1 \xrightarrow{A} V_2 \xrightarrow{B} V_3 \to 0 \) is an exact sequence of linear maps of vector spaces, and let \( W \) be an arbitrary vector space. Prove that the sequence
\[
0 \to V_1 \otimes W \xrightarrow{A \otimes I} V_2 \otimes W \xrightarrow{B \otimes I} V_3 \otimes W \to 0
\]
is exact as well. (In the language of functors and categories, this says that "the tensor multiplication functor is exact". Note that this is true for vector spaces over a field, but \textit{not} for modules over a ring).

### 4. Symmetric and exterior powers

For the definitions of higher symmetric and exterior powers, please see, for example, Sections 5 and 6 in Kostrikin and Manin "Linear Algebra and geometry" (there is full text online available through the library).

(14) Prove that \( \text{Alt}^2 V \cong \land^2 V \).

(15) Prove that
\[
\text{Sym}^m(V \oplus W) = \bigoplus_{a=0}^{m} \text{Sym}^a V \otimes \text{Sym}^{m-a} W;
\]
\[
\land^m(V \oplus W) = \bigoplus_{a=0}^{m} \land^a V \otimes \land^{m-a} W.
\]

(16) Let \( V \) be an \( n \)-dimensional vector space, and \( A : V \to V \) – a linear map. Then \( \land^n V \) is a 1-dimensional vector space, and thus \( \land^n A : \land^n V \to \land^n V \) is multiplication by scalar. Prove that this scalar equals \( \det(A) \).