The space of central functions.


Central functions (class functions) - constant on conjugacy classes.

Remark: Character belongs to \( L^2(G) \), \( \chi_g \in L^2(G) \) for \( g \) rep.

because: \( \chi_g(xg^{-1}) = \text{Tr}(\pi(xg^{-1})g) = \text{Tr}(\pi(g)) = \chi_g(g) \)

\( \text{Tr} \) is conj. invariant.

Theorem: Let \( \{\chi_i\} \) be representations, irreducible of \( G \).

\[ \Rightarrow \chi_i \text{ form an orthonormal basis at } L^2(G) \]

as \( L^2(G) \)

Proof: We already know they are orthogonal, need to show they span \( L^2(G) \).

Lemma: Let \( f \in L^2(G) \). Then \( s(f) : V \rightarrow V \)

\[ s(f) = \frac{1}{|G|} \sum_{g \in G} f(g) \cdot s(g) \]

\( s(f) \) is \( 2 \text{Id}_V \), where \( \lambda = \frac{1}{|G|} \langle f, \chi_i \rangle \).

Proof of theorem (assuming the lemma):

Assume \( \langle f, \chi_i \rangle \neq 0 \)

\[ \Rightarrow \exists \ f \in L^2(G) \text{ such that } \lambda \text{ orthogonal to all } x_i \ (\lambda \text{ non-zero}) \]

Take \( \pi = \text{right regular representation} \).

\[ \pi(f) = \sum_{g \in G} f(g) \chi_i \]

\( \pi(f) \) is a linear operator that acts on each \( V_i \)

by the scalar from the lemma:

\[ \langle f(x), \chi_i \rangle = \langle \pi(f)(x), \chi_i \rangle \]

\[ \Rightarrow \pi(f) \text{ is the } C \text{ operator } \pi(f) : C[G] \rightarrow C[G] \text{ for } \chi_i \rightarrow f \chi_i \]
\[ \Rightarrow \quad \mathcal{Z}(G) = 0 \quad \forall s \in G. \]

**Proof of the Lemma**: An irreducible representation \( s(t) : V \rightarrow V \)

Let's show that if \( \theta \in \mathcal{Z}(G) \), then \( s(g) \) commutes with \( s(t) \) for \( g \in G \).

\[
\begin{align*}
\theta^{-1} s(g^{-1} s(t) s(g) \theta) &= \theta^{-1} s(g^{-1} s(t) s(g) \theta) \\
&= \theta^{-1} \sum_{x \in G} \theta(x) s(g^{-1} x g) \\
&= \theta^{-1} \sum_{x \in G} \theta(x) s(g^{-1} x g) \\
&= \theta^{-1} \sum_{x \in G} \theta(x) \cdot s(g^{-1} x g) \\
&= \theta^{-1} \theta = \theta.
\end{align*}
\]

**Schur's Lemma**

\[ \Rightarrow \quad s(t) = \lambda \cdot \text{Id}. \quad \text{(Note: What is } \lambda^2 \text{?)} \]

\[
\begin{align*}
\text{Tr} \ s(t) &= (\dim V) \cdot \lambda \\
(\mathbf{1}, \chi_\theta) &= \text{Tr} \ s(t) \\
&= (\mathbf{1}, \chi_\theta) = (1, \chi_\theta)_G = \frac{1}{\dim V} \cdot (\mathbf{1}, \chi_{\theta}^G).
\end{align*}
\]

\[ \Rightarrow \quad \lambda = \frac{1}{\dim V} \cdot (\mathbf{1}, \chi_{\theta}^G). \]

**Corollary**: \# of irreps of \( G \) = \# of conjugacy classes in \( G \).

**Proof**: \( \mathcal{Z}(G) \) has two bases:

- characters of irreps \( s \), \( \chi_\theta \).
- characteristic functions of conjugacy classes.

"Fourier transform interchanges these bases."

(at least for abelian groups.)
Algebraic perspective: Using finite groups

Think of $C[G]$ as a ring (with convolution as operation)

$(Z(G)$ is the centre of $C[G]$ (exercise)

$Z(G)$ is even an algebra over $C$.

Def: An algebra $A$ is called simple, if it is not nilpotent.

(i.e. \( \{ a, b : a, b \in A \} = A^2 \neq 0 \)

and has no proper 2-sided ideals.

An algebra $A$ over a field $\mathbb{k}$ is called central if $Z(A) = \mathbb{k}$.

An algebra $A$ is called semisimple if it is a direct sum of simple algebras.

Fact: $A$ semisimple $\iff$ $\text{Rad}(A) = \{ a \in A \text{ nilpotent} \} = 0$.

Key examples.

1. $k$ is any field.

$M_n(k)$ is a central simple algebra.

$\forall n \in \mathbb{N}$.

(Exercise*): 2-sided ideals of $M_n(\mathbb{R})$ are of the form $M_n(J)$ for ideals $J$ in the commutative ring $\mathbb{R}$.

2. $\mathbb{H}$, quaternions over $\mathbb{R}$, division Ring.

simple algebra over $\mathbb{R}$.

Remark: Theorem (by Frobenius). The only division rings over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ up to isomorphism.
Thin: (Waller, Zorn) Suppose $k$ is a field. (no proof)

$\Rightarrow$ Every central and simple algebra over $k$ is of the form $M_n(D)$, where $D$ is a division ring over $k$.

Corollary: if $k$ is closed $k\cap D$ division ring then $D=k$.

Proof: $\alpha \in D$, $k(\alpha)$ = field extension of $k$ = equals $k$.

So over $k$, the only simple, central algebras are $M_n(C)$.

Theorem: (Maeda) (from last time)

$C[G]$ is a semi-simple $C$-algebra: $C[G] = \bigoplus M_{n_i}(C)$

Remark: One can prove this algebraically and then follow Maeda from it.

$C[G] \cong \bigoplus M_{n_i}(C) = \bigoplus \text{End}_V$

$V_i \text{ imps. of } G$.

this is actually commutative (say?)

$Z(G) \longrightarrow C^r$ for $r = \# \text{ of conj. classes of } G$

$= \# \text{ imps.}$

$\gamma \in$ scalar by which $g$ acts on $V_i$, (see lemma)

$\lambda_i = \text{dim}_{V_i}(I, \overline{g})$

Theorem: $g$ irrep of $G$, $\Rightarrow$ dim $g | |G|

Proof idea: (Tate) Note: $\varphi(g)$ is matrix at finite order.

diagonalizable $\Rightarrow$ eigenvalues are roots of 1.

Then $\varphi(g)$ is an algebraic integer.

Then if $z \in Z(G)$ is $\mathbb{F}_p$-valued, then $\sum \varphi(g) \varphi_p(g)$ is also algebraic integer.
Take \( f = x^5 \in \mathbb{Z}[x] \), in integral closure of \( \mathbb{Z} \) in \( \mathbb{Z}(x) \).

\[ \Rightarrow \text{its image in } \mathbb{C}^n \text{ is integral over } \mathbb{Z} . \]

\[ \Rightarrow \text{image at } \alpha_{n} = \frac{[G]}{\dim V} \in \mathbb{C} \text{ has to be integral over } \mathbb{Z} \]

\[ \Rightarrow \text{it didn't divide by } |G| \]

\[ \Rightarrow \frac{|G|}{\dim V} \in \mathbb{Z} \]

"Cool, crazy, efficient proof."

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Upshot: For finite groups: \( n_1, \ldots, n_r = \dim V_1, \ldots, \dim V_r \)

\[ \sum n_i = [G] \]

- Use this if you actually want to find decomposition of a rep of a group.

\[ n_i \mid [G] \]

- \( n = \# \text{ Conjugacy classes of } G \).

Remark: Fact: \( n_i \mid [G] \)

\[ \left( \text{proof uses } \frac{G \times \cdots \times G}{\sim} \right) \]

Later: If \( A \) odd abelian subgroup of \( G \) \[ \Rightarrow n_i \mid \frac{[G]}{[A]} \]

Remark: Representations of \( G \) \[ \Leftrightarrow \mathbb{C}[G] \text{-modules} \]

If \( V \) vector space: \( g : G \to \text{GL}(V) \) \[ \Leftrightarrow \text{action of } \mathbb{C}[G] \text{ on } V \]

\( G \text{ acts by } g(\cdot) \)

\[ k \text{-vector space} \]

\[ (V_i, T) \leftrightarrow k[x]-\text{modules} \]

Exemplar: \( G = \mathbb{Z} \)

\[ \mathbb{C}[G] = \left[ \frac{[\mathbb{Z}]}{\mathbb{Z} \cdot \mathbb{Z}} = \mathbb{Q}[x, z^{-1}] \right] \text{ Laurent polynomials.} \]

\[ \begin{array}{l}
\text{Representations of } \mathbb{Z} \\end{array} \]

\[ 1) \quad G = \mathbb{Z}, \quad \mathbb{C}[G] = \frac{\mathbb{C}[x]}{(x^n - 1)} = \mathbb{C}[\mathbb{Z}/n\mathbb{Z}] \]
Relation with Fourier transform (Harmonic analysis)  

Fourier analysis

Let $f$ be a periodic function on $\mathbb{R}$ with period $T = 2\pi$

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \sin(n \pi x) + b_n \cos(n \pi x)$$

Some assumptions on $f$ (convergence...)

Unit circle with $\mathbb{R} \to S^1 = \mathbb{C}$

Haar-measure (invariant under $S^1$ as group)

$$\mu(g A) = \mu(A)$$

$|\phi| = \text{arc length} \cdot \text{area}$

Consider $\mu_t = 1 \cdot t : S^1 \to \{ z \in \mathbb{C} : 0 < |z| < 1 \}$

Claim: Fourier analysis can be understood as the decomposition at the right regular representations of $S^1$ into irreducible representations.

Most important thing

1) Irreps of $S^1$: $n \in \mathbb{Z}$

From rep-theory

$$e^{i\theta} \mapsto e^{i n \theta}$$

$$e^{i\theta} \mapsto \begin{cases} 1, & \theta \in \mathbb{Z} \\ e^{i \theta}, & \theta \notin \mathbb{Z} \end{cases}$$

This gives us infinitely many 1-dim representations at $S^1$

Indexed by $n \in \mathbb{Z}$

Exercise: For $n \neq 0$ they are not isomorphisms

$$S^1 \to \mathbb{C}, \phi_1 \mapsto \phi$$

$$\phi(n \theta) = \phi_1(\theta)$$

We require $\phi$ to be continuous (as $S^1$ is a topological group)

With this requirement there are all representations.

Proof: 1-dim $\phi(S^1)$ has to be compact $\Rightarrow$ $\phi(S^1) < S^1 \subset \mathbb{C}$

Look at roots of $T$: use continuity, a homomorphism:

Any homom $S^1 \to S^1$ has to be $z \mapsto z^n$ for some $n \in \mathbb{Z}$

More dim reps of $S^1$: $T$ is Nilpotent then it is 1-dim
Fact: A vector space $V$ have a commutative family of linear operators such that each is diagonalizable then they are simultaneously diagonalizable.

$r.e.$ Basis $e_i, \phi(e) \text{ is diagonal}$

Then each $\langle e_i \rangle$ is $S'$-invariant $\Rightarrow$ decomposed in 1-dim.

General fact: Abelian groups have 1-dim irreps.

Back to $S'$: Irreps $\longleftrightarrow n \in \mathbb{Z}$

\[
L^2(S') = \bigoplus_{n \in \mathbb{Z}} C_{\mathbb{C}}
\]

A 1-dim rep, where $S'$ acts by

\[
z \rightarrow z^n.
\]

\[
f = \sum e^{inx}, \phi\text{ basis of } L^2(S') \text{ given by the characters } \chi_n: z \rightarrow z^n.
\]

Then the coefficients of $f$ are $\langle f, \chi_n \rangle_{L^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{inx}) \chi_n(e^{-inx}) \, dx$.

(Fact: $\overline{\chi_n} = \chi_{-n}$)