
(1) **Gauss sums.** Let $p$ be a prime, and let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to S^1$ be a nontrivial character of the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$. We can extend $\chi$ to a function on $\mathbb{Z}/p\mathbb{Z}$ by letting $\chi(0) = 0$. Recall that the group $\mathbb{Z}/p\mathbb{Z}$ is self-dual (this was essentially proved in Homework 1). Consider the Fourier transform of the function $\chi$ on $\mathbb{Z}/p\mathbb{Z}$ (note that we started with a character of the multiplicative group to get this function, but the Fourier transform is happening on the additive group $\mathbb{Z}/p\mathbb{Z}$). The Fourier transform $\hat{\chi} : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ is:

$$\hat{\chi}(x) = \sum_{y \in \mathbb{Z}/p\mathbb{Z}} \chi(y)e^{-2\pi i xy/p} = \sum_{y \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(y)e^{-2\pi i xy/p}.$$ 

Let

$$G(\chi) = \sum_{y \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(y)e^{2\pi iy/p}.$$ 

The sum $G(\chi)$ is called a Gauss sum.

(a) Prove that $\hat{\chi}(x) = \chi(-1)G(\chi)\overline{\chi}(x)$.

(b) Prove that $\overline{G(\chi)} = \chi(-1)G(\overline{\chi})$.

(c) Prove that $|G(\chi)| = \sqrt{p}$ (note that this implies that there are a lot of cancellations in the sum: a naive estimate of its magnitude would be $|G(\chi)| \leq p - 1$, since it’s a sum of $p - 1$ roots of unity).

(2) Let $G$ be a compact group, and let $f \in C(G)$ (where $C(G)$ denotes the space of continuous functions on $G$) be a left and right $G$-finite function (that is, the subspace of $C(G)$ spanned by left and right translates of $f$ is finite-dimensional). Prove that there are only finitely many irreducible representations $(\pi,V)$ of $G$ such that $\pi(f) \neq 0$.

(3) Let $k$ be an arbitrary field. Let $A, B$ be $k$-algebras. An $(A,B)$- bimodule is a $k$-vector space $V$ with both left $A$-module structure and a right $B$-module structure which satisfy $(av)b = a(vb)$ for all $a \in A, b \in B, v \in V$. Note that any left $A$-module is automatically an $(A,k)$-bimodule, and any right $A$-module is a $(k,A)$-bimodule.

Recall that if $V$ is an $(A,B)$-bimodule, and $W$ is a left $B$-module, then one can form the tensor product $V \otimes_B W$ - it is the $k$-vector space

$$(V \otimes_k W)/\langle vb \otimes w - v \otimes bw \mid v \in V, b \in B \rangle,$$

and $V \otimes_B W$ has a left $A$-module structure.

If $A, B, C$ are three $k$-algebras, and if $V$ is an $(A,B)$-bimodule, and $W$ is an $(A,C)$-bimodule, then the vector space $\text{Hom}_A(V,W)$ (this is the space of all left $A$-module homomorphisms from $V$ to $W$) becomes a $(B,C)$-bimodule (in a canonical way) by setting $(hf)(v) = f(vb)$ and $(fc)(v) = f(vc)$ for all $b \in B, f \in \text{Hom}_A(V,W), v \in V$, and $c \in C$.

Now let $A, B, C, D$ be four $k$-algebras, and let $V$ be an $(B,A)$-bimodule, $W$ be a $(C,B)$-bimodule, and $X$ - a $(C,D)$-bimodule. Prove that $\text{Hom}_B(V,\text{Hom}_C(W,X)) \cong \text{Hom}_C(W \otimes_B V, X)$ as $(A,D)$ - bimodules.
Hint: The isomorphism is given by $f \mapsto (w \otimes_B v \mapsto f(v)w)$ for all $v \in V, w \in W,$ and $f \in \text{Hom}_B(V, \text{Hom}_C(W, X))$.

(4) Exercise 6.4 in Serre (p. 50).

(5) For what values of $a, b \in \mathbb{Q}$ is $x = a + b \sqrt{d}$ an algebraic integer, if
(a) $d = 2$
(b) $d = 3$
(c) $d = 5$?