Math 323. Midterm Exam. February 27, 2014. Time: 75 minutes.

(1) [5] Find the centre of the ring of real quaternions $\mathbb{H}$.

(2) [5] Let R be a commutative ring with 1. Prove that if there exists a prime ideal P of R that contains no zero divisors, then R is an integral domain.

(3) Let R be a commutative ring with 1, and I, J – ideals in R.
  (a) [2] Give a sufficient condition for the equality $IJ = I \cap J$ to hold (just a statement, no proof required).
  (b) [3] Give an example of two ideals I and J in a commutative ring R, such that $IJ \neq I \cap J$.
  (c) [6] Prove that if R is a UFD, and $I = (a)$, $J = (b)$ are two principal ideals, then $IJ = I \cap J$ if and only if a and b have no common irreducible factors.

(4) [5] Is 7 prime in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$? If not, factor it as a product of primes, with proof that the factors are prime.

(5) [5] Find an example of an element of $\mathbb{Z}[\sqrt{-3}]$ that is irreducible but not prime (and give a complete proof that it has this property).

(6) [6] Let F be a field that has infinite cardinality. Let n be an arbitrary integer. Prove that for any collection of elements $a_1, \dots a_n \in F$, and any collection of values $c_1, \dots, c_n \in F$ there exists unique polynomial $f \in F[x]$ of degree at most $n-1$ such that $f(a_i) = c_i$ for $1 \leq i \leq n$.

(7) Describe the quotient ring (i.e. find a simpler-looking ring isomorphic to it). Is the ideal $(x^2 + 1)$ maximal in either of these rings?
  (a) [4] $\mathbb{F}_5[x]/(x^2 + 1)$
  (b) [4] $\mathbb{F}_7[x]/(x^2 + 1)$.

(8) [5] Let F be a field, and let $I = (x, y^2 + x^2)$ be the ideal in $F[x, y]$ generated by the polynomials x and $y^2 + x^2$. Describe the quotient ring $F[x, y]/I$ (i.e. find a simpler-looking ring isomorphic to it). (**Hint:** think of $F[x, y]/(x)$ first.)

(9)* (**extra credit, 3 pts**) Suppose we tried to construct a quaternion ring over $\mathbb{F}_p$ by considering expressions $a + bi + cj + dk$ with $a, b, c, d \in \mathbb{F}_p$, and the operations as in the usual quaternion ring (except all the operations with the coefficients are modulo p). Prove that this ring has to contain zero divisors. (Hint: you can quote any theorems proved or even barely mentioned in the course.)