Math 323: Homework 7

1. **Section 13.1 Problem 1.** Show that \( p(x) = x^3 + 9x + 6 \) is irreducible in \( \mathbb{Q}[x] \). Let \( \theta \) be a root of \( p(x) \). Find the inverse of \( 1 + \theta \) in \( \mathbb{Q}(\theta) \).

2. **Section 13.1 Problem 3.** Show that \( x^3 + x + 1 \) is irreducible over \( \mathbb{F}_2 \) and let \( \theta \) be a root. Compute the powers of \( \theta \) in \( \mathbb{F}_2(\theta) \).

3. **Section 9.4 Problem 1.** (b,c,d) Determine whether the following polynomials are irreducible in the rings indicated. For those that are reducible, determine their factorization into irreducibles. The notation denotes the finite field \( \mathbb{Z}/p\mathbb{Z} \), where \( p \) is a prime.
   (a) \( x^3 + x + 1 \) in \( \mathbb{F}_3[x] \)
   (b) \( x^4 + 1 \) in \( \mathbb{F}_5[x] \)
   (c) \( x^4 + 10x^2 + 1 \) in \( \mathbb{Z}[x] \)

4. Prove that the following polynomials are irreducible:
   (a) \( x^9 + 5x^7 - 15x^6 + 30x - 20 \) in \( \mathbb{Q}[x] \).
   (b) \( x^6 - y(y - 1)x^4 + (y - 1)x^3 - (y^2 - 2y + 1)x + y - 1 \) in \( \mathbb{R}[x, y] \).
   (c) \( y^3 - x^2y + xy^2 - 3x \) in \( \mathbb{F}_5[x, y] \).

5. **Section 9.5 Problem 3.** Let \( p \) be an odd prime in \( \mathbb{Z} \) and let \( n \) be a positive integer. Prove that \( x^n - p \) is irreducible over \( \mathbb{Z}[i] \).