Math 323: Homework 6

(1) Prove that the ideals (x) and (x, y) are prime ideals in $\mathbb{Q}[x, y]$ but only the latter ideal is a maximal ideal.

(2) Prove that (x, y) is not a principal ideal in $\mathbb{Q}[x, y]$.

(3) Prove that $F[x, y]/(y^2 - x)$ and $F[x, y]/(y^2 - x^2)$ are not isomorphic as rings for any field F.

(4) Let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$ and let bars denote passage to the quotient $F[x]/(f(x))$. Prove that for each $\overline{g(x)}$ there is a unique polynomial $g_0(x)$ of degree $\leq n - 1$ such that $\overline{g(x)} = \overline{g_0(x)}$ (equivalently, the elements $\overline{1}, \overline{x}, ..., \overline{x^{n-1}}$ are a basis of the vector space $F[x]/(f(x))$ over F – in particular, the dimension of this space is n). **Hint:** Use division algorithm.

(5) Let $f(x)$ be a polynomial in $F[x]$. Prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible.

(6) Exhibit all the ideals in the ring $F[x]/(p(x))$, where F is a field and $p(x)$ is a polynomial in $F[x]$ (describe them in terms of the factorization of $p(x)$).

(7) Determine the gcd of $a(x) = x^3 - 2$ and $b(x) = x + 1$ in $\mathbb{Q}[x]$ and write it as a linear combination of $a(x)$ and $b(x)$.