Math 323: Solutions to Homework 8

Section 10.1. Problem 7. Let $N_1 \subset N_2 \subset \ldots$ be an ascending chain of submodules of $M$. Prove that $\bigcup_{i=1}^\infty N_i$ is a submodule of $N$.

Solution. It is clear that this union is non-empty. Then we just need to check that for any $m, n \in \bigcup_{i=1}^\infty N_i$ and $r \in R$ (where $R$ is the ring our modules are over), we have $m + nr \in \bigcup_{i=1}^\infty N_i$ (see the submodule criterion, Proposition 1 in Section 10.1, p.342). So, let $m, n$ and $r$ be be given. By definition of the union, there exists a positive integer $j$ such that $m, n \in N_j$. Then since $N_j$ is a submodule of $M$, we have $m + nr \in N_j$, and then $m + nr \in \bigcup_{i=1}^\infty N_i$, QED.

Section 10.1 Problem 8. An element $m$ of the $R$-module $M$ is called a torsion element if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\text{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}$$

(a) Prove that if $R$ is an integral domain then $\text{Tor}(M)$ is a submodule of $M$ (called the torsion submodule of $M$)

Solution. Denote the identity element of $M$ as an additive abelian group by 0. Pick any non-zero $r \in R$. Then, $r0 = r(0 + 0) = r0 + r0$. After cancelling $r0$ from both sides, we get $r0 = 0$. Hence, 0 $\in$ Tor(M) and in particular Tor(M) $\neq$ 0. Assume $x, y \in \text{Tor}(M)$ and $r \in R$. Then, by definition, $r_1 x = 0$ and $r_2 y = 0$ for some non-zero $r_1, r_2 \in R$. Using the hypothesis that $R$ is an integral domain, we know $r_1 r_2$ is a non-zero element in $R$. Using commutativity of $R$, we get

$$r_1 r_2 (x + ry) = (r_1 r_2) x + (r_1 r_2) ry = r_2 (r_1 x) + r_1 (r_2 y) = r_2 0 + (r_1 r) 0 = 0 + 0 = 0$$

So $r_1 r_2 (x + ry) = 0$. Since $r_1 r_2$ is non-zero in $R$, it follows that $x + ry \in \text{Tor}(M)$. By the Submodule Criterion (Proposition 1 in page 342), we conclude that Tor(M) is a submodule of $M$.

(b) Give an example of a ring $R$ and a $R$-module $M$ such that Tor(M) is not a submodule.

Solution. Let $R = \mathbb{Z}/10\mathbb{Z}$ and $M = R$ where the action of a ring element on a module element is just the usual multiplication in the ring $R$ (Example 1 in page 338). Note that $\overline{2}$ and $\overline{5}$ are two non-zero elements in $R$, and that $(\overline{2})(\overline{5}) = (\overline{5})(\overline{2}) = \overline{0}$. By definition, $\overline{2}, \overline{5} \in \text{Tor}(M)$. Now let $x = \overline{5}$, $y = \overline{2}$, and $\overline{r} = \overline{3}$. We have just seen $x, y \in \text{Tor}(M)$. However,

$$x + \overline{r}y = \overline{5} + (\overline{3})(\overline{2}) = \overline{11} = \overline{1}$$

For every non-zero $r$ in $R$, we have $r\overline{1} = r \neq 0$, and hence $x + \overline{r}y \notin \text{Tor}(M)$. By Submodule Criterion (Proposition 1 in page 342), we conclude that Tor(M) is not a submodule of $M$.

(c) If $R$ has zero divisors, show that every nonzero $R$-module has nonzero torsion elements.

Solution: We will prove the claim for left $R$-modules. The same statement for right $R$-modules can be similarly proved. Suppose $M$ is a left $R$-module, and that $R$ has a zero divisor, say $a, b \in R$ such that $ba = 0_R$, where $0_R$ denotes the zero of $R$. By definition, $a$ and $b$ are nonzero in $R$. If $am = 0$ for some nonzero $m \in M$, then $a \in \text{Tor}(M)$ and we are done, because we have found a non-zero torsion element
(namely \(m\)). Otherwise, we may assume that \(am \neq 0\) for all non-zero \(m \in M\). Take any non-zero element of \(M\), say \(m'\). Then \(am' \neq 0\), and

\[
b(am') = (ba)m' = 0_{Rm'} = 0
\]

Since \(b\) is non-zero, we get \(am' \in \text{Tor}(M)\) and we are done, since we have found a non-zero torsion element, namely \(am' \in M\).

**Remark:** In the solution, we used the fact that \(0_{Rm} = 0\) for \(m \in M\). Here \(0_R\) denotes the zero of \(R\), and 0 denotes the identity element of \(M\) as an additive abelian group. This is easily proven from the module axioms: \(0_{Rm} = (0_R + 0_R)m = 0_Rm + 0_Rm\). Using cancellation in the additive group \(M\), we obtain \(0_{Rm} = 0\).

**Section 10.1 Problem 9:** If \(N\) is a submodule of \(M\), the annihilator of \(N\) in \(R\) is defined to be \(\{r \in R \mid rn = 0\ \text{for all} \ n \in N\}\). Prove that the annihilator of \(N\) in \(R\) is a 2-sided ideal of \(R\).

**Solution:** Let \(I_N = \{r \in R \mid rn = 0\ \text{for all} \ n \in N\}\). We want to show that \(I_N\) is a 2-sided ideal of \(R\). Since \(0n = 0\) for all \(n \in N\), it follows that \(0 \in I_N\), so in particular \(I_N \neq \emptyset\). Given \(a, b \in I_N\), we observe that for each \(n \in N\),

\[
(a - b)n = an - bn = 0 - 0 = 0
\]

so that \(a - b \in I_N\), and \(I_N\) is closed under subtraction. Finally, given \(r \in R\) and \(a \in I_N\), we have

\[
(ra)(n) = r(an) = r0 = 0 \quad \text{and} \quad (ar)(n) = a(rn) = 0
\]

for each \(n \in N\). Thus, \(ar \in I_N\) and \(ra \in I_N\). We conclude that \(I_N\) is a two-sided ideal in \(R\).

**Section 10.1 Problem 10:** If \(I\) is a right ideal of \(R\), the annihilator of \(I\) in \(M\) is defined to be \(\{m \in M \mid am = 0\ \text{for all} \ a \in I\}\). Prove that the annihilator of \(I\) in \(M\) is a submodule of \(M\).

**Solution:** Let \(N_I = \{m \in M \mid am = 0\ \text{for all} \ a \in I\}\). We want to show that \(N_I\) is a submodule of \(M\). Since \(a0 = 0\) for all \(a \in I\), it follows that \(0 \in N_I\), so in particular \(N_I \neq \emptyset\). Given \(x, y \in N_I\) and \(r \in R\), we obtain that for each \(a \in I\), we have \(ar \in I\) (since \(I\) is an ideal), and so

\[
a(x + ry) = ax + (ar)y = 0 + 0 = 0
\]

Hence, \(x + ry \in N_I\). By Submodule Criterion, \(N_I\) is a submodule of \(M\).

**Section 10.1 Problem 11.** Let \(M\) be the abelian group (i.e. a \(\mathbb{Z}\)-module) \(\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/50\mathbb{Z}\).

(a) Find the annihilator of \(M\) in \(\mathbb{Z}\) (i.e. a generator for this principal ideal).

**Solution:** We want to determine all \(r \in \mathbb{Z}\) such that \(rm = 0\) for all \(m \in M\). We can write \(m = (\overline{x}, \overline{y}, \overline{z})\) where \(\overline{x} \in \mathbb{Z}/24\mathbb{Z}\), \(\overline{y} \in \mathbb{Z}/15\mathbb{Z}\) and \(\overline{z} \in \mathbb{Z}/50\mathbb{Z}\). The condition \(rm = 0\) now gives \(r(\overline{x}, \overline{y}, \overline{z}) = (0, 0, 0)\), meaning that \(r\overline{x} = 0\), \(r\overline{y} = 0\) and \(r\overline{z} = 0\). For this to be true for all \(m = (\overline{x}, \overline{y}, \overline{z})\), we need \(r\) to be the least common multiple of 24, 15, 50. Since \(\text{lcm}(24, 15, 50) = 600\), we conclude that the annihilator of \(M\) in \(\mathbb{Z}\) is \((600)\), the principal ideal generated by \(600 \in \mathbb{Z}\).
(b) Let \( I = 2\mathbb{Z} \). Describe the annihilator of \( I \) in \( M \) as a direct product of cyclic groups.

**Solution:** We are looking for all \( m \in M \) such that \( am = 0 \) for all \( a \in 2\mathbb{Z} \). It is enough to determine all \( m \in M \) such that \( 2m = 0 \) (because \( 2\mathbb{Z} \) is principal). Write \( m = (x, y, z) \) where \( x \in \mathbb{Z}/24\mathbb{Z}, \ y \in \mathbb{Z}/15\mathbb{Z} \) and \( z \in \mathbb{Z}/50\mathbb{Z} \).

The condition \( 2m = 0 \) now gives \( 2(x, y, z) = (0, 0, 0) \), meaning that \( 2x = 0, 2y = 0 \) and \( 2z = 0 \). Or equivalently,

\[
\begin{align*}
2x &= 0 \pmod{24} \\
2y &= 0 \pmod{15} \\
2z &= 0 \pmod{50}
\end{align*}
\]

By rules governing modular arithmetic, the above system is equivalent to

\[
\begin{align*}
x &= 0 \pmod{12} \\
y &= 0 \pmod{15} \\
z &= 0 \pmod{25}
\end{align*}
\]

Hence, \( m = (x, y, z) \in M \) is in the annihilator of \( 2\mathbb{Z} \) if and only if the above system is satisfied. Going back to the quotient group, we see that \( m = (x, y, z) \in M \) is in the annihilator of \( 2\mathbb{Z} \) if and only if

\[
m \in 12\mathbb{Z}/24\mathbb{Z} \times 15\mathbb{Z}/15\mathbb{Z} \times 25\mathbb{Z}/50\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}
\]

Hence, the annihilator of \( 2\mathbb{Z} \) in \( M \) is \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) which is isomorphic to simply \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

**Section 10.1 Problem 20.** Let \( F = \mathbb{R} \), let \( V = \mathbb{R}^2 \) and let \( T \) be the linear transformation from \( V \) to \( V \) which is rotation clockwise about the origin by \( \pi \) radians. Show that every subspace of \( V \) is an \( F[x] \)-submodule for this \( T \).

**Solution:** We recall that a subspace \( U \subset V \) is called \( T \)-stable if \( T(U) \subset U \). As shown in page 341, \( F[x] \)-submodules of \( V \) are precisely the \( T \)-stable subspaces of \( V \). Hence, we need to prove that every subspace of \( V \) is \( T \)-stable for this particular linear transformation \( T \) (where \( T \) is rotation clockwise about the origin by \( \pi \) radians). We can write \( T \) as

\[
T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} \cos(-\pi) & -\sin(-\pi) \\ \sin(-\pi) & \cos(-\pi) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} = -1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

So \( T \) is simply scaling the vector by \(-1\), i.e. \( T(v) = -v \) for each \( v \in V \). Consider any subspace \( U \subset V \). Let \( u \in U \). Then, \( T(u) = -u \in U \) since \( U \) is a subspace (and therefore closed under scalar multiplication). Since \( u \) was arbitrary, it follows that \( T(U) \subset U \), implying that \( U \) is a \( T \)-stable subspace of \( V \), and so \( U \) is \( F[x] \)-submodule of \( V \). Since this is true for every subspace \( U \subset V \), the proof is complete.

**Section 10.2 Problem 8.** Let \( \varphi : M \to N \) be an \( R \)-module homomorphism. Prove that \( \varphi(\text{Tor}(M)) \subset \text{Tor}(N) \).

**Solution:** We recall the definition of torsion submodule:

\( \text{Tor}(M) = \{ m \in M \mid rm = 0 \text{ for some nonzero } r \in R \} \)
Suppose $x \in \varphi(\text{Tor}(M))$. By definition, $x = \varphi(m)$ for some $m \in \text{Tor}(M)$, meaning that $rm = 0$ for some nonzero $r \in R$. Using the fact that $\varphi$ is a $R$-module homomorphism, we obtain

$$0 = \varphi(0) = \varphi(rm) = r\varphi(m) = rx$$

Thus $rx = 0$. Note that $x = \varphi(m) \in N$. Since $r$ is nonzero element in $R$, from $rx = 0$ we immediately get $x \in \text{Tor}(N)$. As $x$ was arbitrary element of $\varphi(\text{Tor}(M))$, we deduce that $\varphi(\text{Tor}(M)) \subset \text{Tor}(N)$. 