Math 323: Homework 2 solutions

Section 7.2 Problem 3. Define the set $R[[x]]$ of formal power series in the indeterminate $x$ with coefficients from $R$ to be all formal infinite sums

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Define addition and multiplication of power series in the same way as for power series with real or complex coefficients i.e. extend polynomial addition and multiplication to power series as though they were “polynomials of infinite degree”

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \times \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$

(a) Prove that $1 - x$ is a unit in $R[[x]]$ with inverse $1 + x + x^2 + \ldots$

Solution: We recall that the multiplicative identity in $R[[x]]$ is power series of the form $\sum_{n=0}^{\infty} d_n$ where $d_0 = 1$ and $d_n = 0$ for each $n \geq 1$. Now, if we formally multiply the following two quantities, we exactly get

$$(1 - x)(1 + x + x^2 + x^3 + \ldots) = 1 + x - x + x^2 - x^2 + x^3 - x^3 + \ldots = \sum_{n=0}^{\infty} d_n x^n$$

one can also check

$$(1 + x + x^2 + x^3 + \ldots)(1 - x) = 1 + x - x + x^2 - x^2 + x^3 - x^3 + \ldots = \sum_{n=0}^{\infty} d_n x^n$$

meaning that $(1 - x)$ is a unit in $R[[x]]$ with inverse $1 + x + x^2 + \ldots$, as desired.

(b) Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$ if and only if $a_0$ is a unit in $R$.

Solution: Since the multiplicative identity in $R[[x]]$ is $\sum_{n=0}^{\infty} d_n$ with $d_0 = 1$, $d_n = 0$ for $n \geq 1$, if we assume $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$, then

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \times \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} d_n x^n$$

Or equivalently,

$$\sum_{n=0}^{\infty} \left(\sum_{i+j=n} a_i b_j\right) x^n = \sum_{n=0}^{\infty} d_n x^n$$

Matching the coefficients of power of $x^0$, we see that $a_0 b_0 = d_0 = 1$, so that $a_0$ is a unit in $R$. This proves one direction.

For the converse, assume $a_0$ is a unit. Then there exists a $c_0$ with $a_0 c_0 = 1$. Define the power series $\sum_{n=0}^{\infty} b_n x^n$ as follows. Let $b_0 = c_0$, and inductively define

$$b_{k+1} = -b_0(a_1 b_k + a_2 b_{k-1} + \cdots + a_k b_0)$$
We then verify that for each \( n \geq 1 \) we get (using definition of \( b_n \))

\[
\sum_{i+j=n} a_ib_j = a_0b_n + (a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0)
\]

\[= -a_0b_0(a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0) + (a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0)
\]

\[= -a_0b_0(a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0) + (a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0)
\]

\[= 0
\]

Therefore, we get that

\[
\left( \sum_{n=0}^{\infty} a_n x^n \right) \times \left( \sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} \left( \sum_{i+j=n} a_ib_j \right) x^n
\]

\[a_0b_0 + \sum_{n=1}^{\infty} \left( \sum_{i+j=n} a_ib_j \right) x^n = a_0b_0 = 1
\]

Hence, \( \sum_{n=0}^{\infty} a_n x^n \) has a right inverse. If we assume \( R \) is commutative, then it automatically follows that \( \sum_{n=0}^{\infty} a_n \) is a unit. If \( R \) is not commutative, one can also establish a left inverse as follows: let \( b'_0 = c_0 \) (recall that \( c_0 \) is two-sided inverse of \( a_0 \), i.e. \( c_0a_0 = a_0c_0 = 1 \)), and inductively define

\[b'_k + 1 = -(a_1b'_k + a_2b'_{k-1} + \cdots + a_{k+1}b'_0)b_0
\]

then one easily checks that \( \sum_{n=0}^{\infty} b'_n x^n \) is a left inverse for \( \sum_{n=0}^{\infty} a_n x^n \). By Problem 28(a) from Section 7.1 to conclude that \( \sum_{n=0}^{\infty} a_n x^n \) is a unit in \( R[[x]] \).

**Section 7.3 Problem 17.** Let \( R \) and \( S \) be nonzero rings with identity and denote their respective identities by \( 1_R \) and \( 1_S \). Let \( \phi : R \to S \) be a nonzero homomorphism of rings.

(a) Prove that if \( \phi(1_R) \neq 1_S \), then \( \phi(1_R) \) is a zero divisor in \( S \). Deduce that if \( S \) is an integral domain, then every non-zero ring homomorphism from \( R \) to \( S \) sends the identity of \( R \) to the identity of \( S \).

*Solution:* If \( \phi(1_R) = 0 \), then for each \( r \in R \), we get \( \phi(r) = \phi(r \cdot 1_R) = \phi(r)\phi(1_R) = 0 \), so \( \phi(r) = 0 \) for all \( r \in R \), which contradicts the assumption that \( \phi \) is a non-zero homomorphism. Thus, \( \phi(1_R) \neq 0 \). Now, we assume \( \phi(1_R) \neq 1_S \), and observe that

\[\phi(1_R) = \phi(1_R \cdot 1_R) = \phi(1_R)\phi(1_R)\]

which leads to \( \phi(1_R) = \phi(1_R)\phi(1_R) = 0 \), or equivalently, \( \phi(1_R)(1_S - \phi(1_R)) = 0 \). Since \( \phi(1_R) \neq 0 \) and \( 1_S - \phi(1_R) \neq 0 \) (because \( \phi(1_R) \neq 1_S \)) it follows that \( 1_R \) is a left zero divisor. But also, the same equation above leads to \( \phi(1_R)\phi(1_R) - \phi(1_R) = 0 \), or equivalently, \( (\phi(1_R) - 1_S)\phi(1_R) = 0 \) which implies \( \phi(1_R) \) is a right zero divisor. Combining these two facts, we see that \( 1_R \) is a zero divisor, as desired. If \( S \) is an integral domain, then \( S \) has no zero divisors, and we deduce that \( \phi(1_R) = 1_S \) for every non-zero homomorphism \( \phi \) from \( R \) to \( S \).

(b) Prove that if \( \phi(1_R) = 1_S \) then \( \phi(u) \) is a unit in \( S \), and that \( \phi(u^{-1}) = \phi(u)^{-1} \) for each unit \( u \in R \).
Solution: Assume $u \in R$ is a unit. Then, by definition, there exists $t \in R$ such that $ut = tu = 1_R$. Since we are assuming $\phi(1_R) = 1_S$, applying $\phi$ to this equation we get

$$\phi(ut) = \phi(tu) = \phi(1_R) = 1_S \Rightarrow \phi(u)\phi(t) = \phi(t)\phi(u) = 1_S$$

Therefore, $\phi(u)$ is a unit in $S$, and its inverse is $\phi(t)$. But by definition, $t = u^{-1}$, so inverse of $\phi(u)$ is $\phi(u^{-1}) = (\phi(u))^{-1}$.

Section 7.3 Problem 2. Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.

Solution: Assume, to the contrary, that $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are isomorphic rings. Then, there exists an isomorphism $\psi : \mathbb{Q}[x] \to \mathbb{Z}[x]$. We first claim that $\psi(1) = 1$ (where 1 indicates constant polynomial 1 in both rings). To prove this, observe that since $\psi$ is surjective (as it is an isomorphism), there exists $b \in \mathbb{Q}[x]$ with $\psi(b) = 1$.

But then, observe that

$$\psi(1) = \psi(1) \cdot 1 = \psi(1)\psi(b) = \psi(1 \cdot b) = \psi(1) = 1$$

So $\psi(1) = 1$, as claimed. Also,

$$\psi(2)\psi\left(\frac{1}{2}\right) = \psi\left(2 \cdot \frac{1}{2}\right) = \psi(1) = 1 \quad (*)$$

By definition, $\psi(2) \in \mathbb{Z}[x]$ and $\psi\left(\frac{1}{2}\right) \in \mathbb{Z}[x]$. Assume that $\psi(2)$ has degree $d_1$, and $\psi\left(\frac{1}{2}\right)$ has degree $d_2$. So polynomial $\psi(2)\psi\left(\frac{1}{2}\right)$ has degree $d_1 + d_2$, and so the equation $(*)$ yields that $d_1 + d_2 = 0$ (because the constant polynomial 1 has degree 0). Since degrees are non-negative, we conclude that $d_1 = d_2 = 0$, so that $\psi(2)$ and $\psi\left(\frac{1}{2}\right)$ are constant polynomials in $\mathbb{Z}[x]$. Thus, $\psi(2), \psi\left(\frac{1}{2}\right) \in \mathbb{Z}$. So equation $(*)$ says that two integers multiply to get 1. This is only possible if either

$$\psi(2) = \psi\left(\frac{1}{2}\right) = 1$$

or

$$\psi(2) = \psi\left(\frac{1}{2}\right) = -1$$

In either case, we see that $\psi$ is not injective (because it maps 2 and $\frac{1}{2}$ to same value), and this is a contradiction because we are assuming that $\psi$ is isomorphism (which necessarily makes it injective).

Note that there were many other ways to solve this problem: for example, look at the units in both rings.

Section 7.3. Problem 6. Decide which of the following are ring homomorphisms from $M_2(\mathbb{Z})$ to $\mathbb{Z}$:

(a) \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) $\mapsto$ a (projection onto the 1, 1 entry)

We claim that this map is not a ring homomorphism. Let $f$ represent this map. Then,

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cdot f\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}\right) = ax$$

while

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix}\right) = f\left(\begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix}\right) = ax + bz$$
Since $ax \neq ax + bz$ in general, we see that $f$ **fails** to satisfy the multiplicative property of ring a homomorphism (for example when $b = z = 1$).

(b) $(a \ b \ c \ d) \mapsto a + d$ (the trace of the matrix)

We claim that this map is **not** a ring homomorphism. Let $f$ represent this map. Then,

$$f \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \cdot f \left( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = (a + d)(x + w) = ax + aw + dx + dw$$

while

$$f \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = f \left( \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix} \right) = ax + bz + cy + dw$$

Since $ax + aw + dx + dw \neq ax + bz + cy + dw$ in general, since $aw + dx \neq bz + cy$ (for example, when $a = w = 1$ and $d = x = b = z = c = y = 0$), we see that $f$ **fails** to satisfy the multiplicative property of a ring homomorphism.

(b) $(a \ b \ c \ d) \mapsto ad - bc$ (the determinant of the matrix)

We claim that this map is **not** a ring homomorphism. Let $f$ represent this map. Then,

$$f \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + f \left( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = (ad - bc) + (xw - yz)$$

while

$$f \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + f \left( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = f \left( \begin{pmatrix} a + x & b + y \\ c + z & d + w \end{pmatrix} \right) = (a + x)(d + w) - (b + y)(c + z)$$

$$= (ad - bc) + (xw - yz) + (aw + xz - bz - yc)$$

Since $(ad - bc) + (xw - yz) \neq (ad - bc) + (xw - yz) + (aw + xz - bz - yc)$ in general, since $aw + xz - bz - yc \neq 0$ in general (for example when $a = w = 1$ and $x = d = b = z = y = c = 0$), we see that $f$ **fails** to satisfy additive property of a ring homomorphism.

**Section 7.3 Problem 10.** Decide which of the following are ideals of $\mathbb{Z}[x]$:

(a) the set of all polynomials whose constant term is a multiple of 3.

This set $I$ is an ideal. It is clearly non-empty, as $0 \in I$. Given $p(x), q(x) \in I$, the constant term of the polynomial $p(x) - q(x)$ is the sum of the constant terms of $p(x)$ and $q(x)$, so is divisible by 3 (as the difference of two numbers divisible by 3 is again divisible by 3), implying $p(x) - q(x) \in I$, which implies that $I$ is a group under addition. Finally, given $p(x) \in I$, and $r(x) \in \mathbb{Z}[x]$, the constant term of $p(x)q(x)$ is product of constant terms of $p(x)$ and $r(x)$, and so is divisible by 3 (as product of a number divisible by 3 by any other integer is divisible by 3), implying $p(x)r(x) \in I$.

(b) the set of all polynomials whose coefficient of $x^2$ is a multiple of 3.

This set $I$ is **not** an ideal, because $x + 1 \in I$ (because the coefficient of $x^2$ is 0), but $(x + 1)^2 = x^2 + 2x + 1 \notin I$ because 3 does not divide 1. So $I$ is not closed under multiplication, so cannot be an ideal.
We consider the evaluation map \( f \).

This set \( I \) is an ideal. It is clear that \( I \) is nonempty, as \( 0 \in I \). Given \( p(x), q(x) \in I \), write \( p(x) = a_nx^n + \cdots + a_3x^3 \) and \( q(x) = b_nx^n + \cdots + b_3x^3 \). Then, \( p(x) - q(x) = (a_n-b_n)x^n + \cdots + (a_3-b_3)x^3 \in I \) by definition. Also, given \( r(x) \in \mathbb{Z}[x] \), we see that the constant term, coefficient of \( x \) and \( x^2 \) in \( p(x)r(x) = x^3(a_nx^{-3} + \cdots + a_3)r(x) \) are all zero, (because a factor of \( x^3 \) is present), so \( p(x)r(x) \in I \). Note: This is the ideal generated by the polynomial \( x^3 \).

(d) \( \mathbb{Z}[x^2] \) (i.e. the polynomials in which only even powers of \( x \) appear). The set \( \mathbb{Z}[x^2] \) is not an ideal. We observe that \( x^2 \in \mathbb{Z}[x^2] \), but \( x(x^2) = x^3 \notin \mathbb{Z}[x^2] \), so \( \mathbb{Z}[x^2] \) is not closed under multiplication by an element of the ring \( \mathbb{Z}[x] \). Thus, \( \mathbb{Z}[x^2] \) is not an ideal.

(e) the set of polynomials whose coefficients sum to zero.

This set \( I \) is an ideal. The sum of coefficients in a given polynomial \( f(x) \in \mathbb{Z}[x] \) is simply \( f(1) \). So, we want to show that \( I = \{ f(x) \in \mathbb{Z}[x] : f(1) = 0 \} \) is an ideal. We consider the evaluation map \( \psi: \mathbb{Z}[x] \rightarrow \mathbb{Z} \) defined by \( \psi(f(x)) = f(1) \). In class, we showed that \( \psi \) is a homomorphism. But \( \ker(\psi) = I = \{ f(x) \in \mathbb{Z}[x] : f(1) = 0 \} \). Since kernel of a ring homomorphism is an ideal, we deduce that \( I \) is an ideal. (This is the ideal generated by \( x-1 \).)

(f) the set of polynomials \( p(x) \) such that \( p'(0) = 0 \).

This set \( I \) is not an ideal. We observe that \( x^2 + 1 \in I \), because \( (x^2 + 1)' = 2x \), and \( 2x \) evaluated at 0 is simply 0. But \( x(x^2 + 1) = x^3 + x \notin I \), because derivative of \( x^3 + x \) is \( 3x^2 + 1 \), whose value at 0 is 1, not zero. Thus, \( I \) is not closed under multiplication by an element in \( \mathbb{Z}[x] \). We conclude that \( I \) not is an ideal.

**Section 7.3 Problem 14.** Prove that the ring \( M_4(\mathbb{R}) \) contains a subring that is isomorphic to the real Hamilton Quaternions \( \mathbb{H} \).

**Solution:** We let

\[
\begin{align*}
\text{Id} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}, \\
I &= \begin{pmatrix} 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \end{pmatrix}, \\
J &= \begin{pmatrix} 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \end{pmatrix}, \\
K &= \begin{pmatrix} 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

It is easily checked that these matrices satisfy:

\[
I^2 = J^2 = K^2 = -\text{Id}, \quad IJ = -K = -JI, \quad JK = I = -KJ, \quad KI = J = -IK.
\]

These relations are exactly those which characterize real Hamiltonian Quaternions \( \mathbb{H} \). So we let \( S = \{ a\text{Id} + bi + cj + dK : a, b, c, d \in \mathbb{R} \} \subset M_4(\mathbb{R}) \). One can write down what the elements of \( S \) look like, explicitly:

\[
S = \left\{ \begin{pmatrix} a & b & c & d \\
-b & a & -d & c \\
-c & d & a & -b \\
-d & -c & b & a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}
\]

Then, a homomorphism \( \psi: \mathbb{H} \rightarrow S \) defined by

\[
\psi(a + bi + cj + dk) = a \text{Id} + bi + cj + dK
\]
is the desired isomorphism. The fact that it is a ring homomorphism follows by definition: it respects multiplication on the set \( \{1, i, j, k\} \), and then it is a homomorphism on the rest of \( \mathbb{H} \) by linearity (make sure you understand this statement!); it is clearly a surjective map onto the set \( S \). The only thing one needs to check is that this is an injective map from \( \mathbb{H} \) to \( M_4(\mathbb{C}) \). Recall that a homomorphism is injective if its kernel is trivial, and the kernel is a 2-sided ideal. Since \( \mathbb{H} \) is a division ring, it has no proper two-sided ideals, and since \( \psi \) is a non-zero homomorphism, we are done.

**Section 7.3 Problem 19.** Prove that if \( I_1 \subset I_2 \subset \cdots \) are ideals of \( R \), then \( \bigcup_{n=1}^\infty I_n \) is an ideal in \( R \).

*Solution:* Let \( I = \bigcup_{n=1}^\infty I_n \). We want to show that \( I \) is an ideal. First, as \( I_1 \) is an ideal, we get \( 0 \in I_1 \subset I \) and so \( I \) is non-empty. Assume \( a, b \in I \). Then, by the definition of set-theoretic union, there exists \( m_1, m_2 \in \mathbb{N} \) with \( a \in I_{m_1} \) and \( b \in I_{m_2} \).

Now, let \( r \in R \) be any element. Then, since \( a \in I_{m_1} \) and \( I_{m_1} \) is an ideal, we get that \( ra \in I_{m_1} \subset I_n = I \), so \( ra \in I \). This shows that \( I \) is closed under multiplication.

Next, we claim that \( I \) is closed under addition. Let \( a, b \in I \). Then, since the chain of ideals are ascending, we get that \( I_{m_1} \subset I_m \), and \( I_{m_2} \subset I_m \), which gives \( a, b \in I_m \). Since \( I_m \) is an ideal, we see that \( a + b \in I_m \subset \bigcup_{n=1}^\infty I_n = I \). Thus, \( I \) is closed under addition. Hence, \( I \) is an ideal.

**Section 7.3 Problem 21.** Prove that all the two-sided ideals in the matrix ring \( M_n(R) \), where \( R \) is a ring with 1, are of the form \( M_n(J) \), where \( J \subset R \) is a two-sided ideal.

*Solution.* Here’s a short "skeleton" of the solution, without details. First, let us check that \( M_n(J) \) is an ideal. It is obviously closed under addition and subtraction; we just need to check that if \( A \in M_n(J) \), then for all \( B \in M_n(R) \), \( AB \) and \( BA \) lie in \( M_n(J) \). Consider the matrix \( AB \). All its entries are of the form \( c_{mk} \sum_{i=1}^n a_{mi} b_{ik} \); each of the terms \( a_{mi} b_{ik} \) is in \( J \) since \( J \) is an ideal of \( R \), and so \( c_{mk} \in J \). A similar argument works for \( BA \).

The hard part is the converse – that all the ideals are of this form. Let \( I \) be a two-sided ideal of \( M_n(R) \). Let \( J \) be the two-sided ideal in \( R \) generated by all the matrix entries of all the elements of \( I \) (that is, for every \( A = (a_{ij}) \in I \), take all the entries \( a_{ij} \) and put them into this big bag of generators of \( J \)). Then we claim that \( I = M_n(J) \).

The inclusion \( I \subset M_n(J) \) is clear: by definition of \( J \), all the entries of all elements of \( I \) are in \( J \). So we just need to prove that \( I \supset M_n(J) \).

Fix four numbers \( i, j, k, l \in \{0, \ldots, n\} \). Given a matrix \( A = (a_{ij}) \), denote by \( S_{i,j}^k,l(A) \) the matrix whose \( k,l \)-th entry is the same as the \( j,i \)-th entry \( a_{ji} \) of \( A \), and all other entries are zero. It is not hard to prove (skipped) that the two-sided ideal \( M_n(J) \) is generated (as a two-sided ideal in \( M_n(R) \)) by the set \( \{ S_{i,j}^k,l(A) \mid 1 \leq i, j, k, l \leq n, A \in I \} \).

Then to show that \( I \supset M_n(J) \), it suffices to show that \( I \) contains all the elements \( S_{i,j}^k,l(A) \) for \( A \in I \). It remains to observe that \( S_{i,j}^k,l(A) \) is, in fact, the product \( e_{k,l} A e_{j,i} \) (unless I am confusing the indices), where \( e_{k,l} \) is the matrix whose \( k,l \)-th entry is 1 and all other entries are zero. Hence, \( S_{i,j}^k,l(A) \) lies in \( I \).

Comment: this solution is an example of an argument where you have an ideal defined by some property, and you need to construct a convenient set of generators for this ideal (here, we define \( M_n(J) \) to be the ideal of matrices whose entries lie in
J, and look at a set of generators that consists of matrices with just one non-zero
entry).

Section 7.3 Problem 24. Let \( \phi : R \to S \) be a ring homomorphism.

1. Prove that if \( J \subset S \) is an ideal, then \( \phi^{-1}(J) \subset R \) is an ideal. Apply this
to the special case when \( R \) is a subring of \( S \) and \( \phi \) is the inclusion map to
deduce that if \( J \subset S \) is an ideal and \( R \subset S \) a subring, then \( J \cap R \subset R \) is
an ideal.

\[ \text{Solution:} \quad \text{Since } 0 \in J, \text{ and } \phi(0) = 0 \text{ we see that } 0 \in \phi^{-1}(J) \text{ and so}
\phi^{-1}(J) \text{ is nonempty. Assume } a, b \in \phi^{-1}(J). \text{ Then, there exist } c, d \in J \text{ with}
\phi(a) = c \text{ and } \phi(b) = d. \text{ Since } J \text{ is an ideal, we have } c + d \in J. \text{ Now, we have}
\phi(a + b) = \phi(a) + \phi(b) = c + d \in J
\]
which gives \( a + b \in \phi^{-1}(J) \), so that \( \phi^{-1}(J) \) is closed under addition. Similarly,
given any element \( r \in R \) we observe that
\phi(ra) = \phi(r)\phi(a) = \phi(r)c \in J \quad \phi(ar) = \phi(a)\phi(r) = c\phi(r) \in J
as \( J \) is an ideal. The above two equations shows \( ar \in \phi^{-1}(J) \) and \( ra \in \phi^{-1}(J) \).
We conclude that \( \phi^{-1}(J) \) is an ideal in \( R \).

Now, consider the special case when \( R \) is a subring of \( S \), and \( \phi \) is the
inclusion homomorphism. Then if \( J \subset S \) is an ideal, from above we know
that \( \phi^{-1}(J) \) is an ideal in \( R \). Since the map is the inclusion map, we get
\( \phi^{-1}(J) = J \cap R \). So \( J \cap R \) is an ideal in \( R \).

2. Prove that if \( \phi \) is surjective and \( I \subset R \) is an ideal, then \( \phi(I) \subset S \) is an
ideal. Give an example where this fails if \( \phi \) is not surjective.

\[ \text{Solution:} \quad \text{The proof that } \phi(I) \text{ is closed under addition is as usual.}
\]
Now, we need to show that for any \( a \in I \) and \( s \in S \), \( sa \in \phi(I) \). Note
that surjectivity is used here in a crucial way: there exists \( r \in R \), such that
\( \phi(r) = s \), and \( b \in I \), such that \( \phi(b) = a \). Then
\[ sa = \phi(r)\phi(b) = \phi(rb) \in \phi(I), \]
since \( rb \in I \) because \( I \) is an ideal.

Without the surjectivity assumption, the statement fails: consider, for
example, the inclusion map \( \phi : \mathbb{Z} \to \mathbb{Q} \). The only ideals in \( \mathbb{Q} \) are zero and
\( \mathbb{Q} \) (since \( \mathbb{Q} \) is a field), and so the image of any non-zero ideal in \( \mathbb{Z} \) is not an
ideal.

10. Let \( \mathbb{Q}[\pi] \) be the set of all real numbers of the form
\[ r_0 + r_1\pi + \cdots + r_n\pi^n \text{ with } r_i \in \mathbb{Q}, n \geq 0. \]
It is easy to show that \( \mathbb{Q}[\pi] \) is a subring or \( \mathbb{R} \) (you don’t have to write the proof of
this). Is \( \mathbb{Q}[\pi] \) isomorphic to the polynomial ring \( \mathbb{Q}[x] ? \)

\[ \text{Solution:} \quad \text{We claim that } \mathbb{Q}[\pi] \text{ is isomorphic to } \mathbb{Q}[x]. \text{ To prove this, we consider}
the evaluation homomorphism } \psi : \mathbb{Q}[x] \rightarrow \mathbb{Q}[\pi] \text{ defined by}
\psi(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0) = a_n\pi^n + a_{n-1}\pi^{n-1} + \cdots + a_0\]
Now, we claim that kernel of this map is $\ker(\psi) = \{0\}$. Assume not. Then, there exists a non\textit{zero} polynomial $f(x) \in \ker(\psi)$ with $f(x) \in \mathbb{Q}[x]$. By definition of $\psi$, this would imply that $f(\pi) = 0$. So $\pi$ is a root of some nonzero polynomial with rational coefficients. This is a contradiction because $\pi$ is transcendental. We conclude that $\ker(\psi) = \{0\}$. By First Isomorphism Theorem, we obtain that

$$\mathbb{Q}[x]/(0) \cong \mathbb{Q}[\pi] \iff \mathbb{Q}[x] \cong \mathbb{Q}[\pi]$$

as desired.