Math 323: Homework 10 – the last one! Due Tuesday April 4.

1. Section 10.2, Problem 11. Let $A_1, A_2, ..., A_n$ be R-modules and let B_i be a submodule of A_i for each $i = 1, 2, ..., n$. Prove that

$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \cong (A_1/B_1) \times \cdots \times (A_n/B_n)$$

2. Section 10.2 Problem 12. Let I be a left ideal of R and let n be a positive integer. Prove

$$R^n/IR^n \cong R/IR \times \cdots \times R/IR$$

3. Section 10.3 Problem 2. Assume R is commutative. Prove that $R^n \cong R^m$ if and only if $n = m$, i.e. two free R-modules of finite rank are isomorphic if and only if they have the same rank.

4. Section 10.3 Problem 6. Prove that if M is a finitely generated R-module that is generated by n elements then every quotient of M may be generated by n (or fewer) elements. Deduce that quotients of cyclic modules are cyclic.

5. Section 10.3 Problem 9. An R-module M is called *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M. Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as a generator. Determine all the irreducible $\mathbb{Z}$-modules.

6. Section 10.3 Problem 23. Show that a direct sum of free R-modules is free.

Problem 7.

(a) Prove that two vectors $\bar{v}_1$, $\bar{v}_2$ form a basis of the module $\mathbb{Z}^2$ if and only if the parallelogram spanned by them contains no lattice points. Suppose that $\bar{v}_1 = \langle a_1, b_1 \rangle$ and $\bar{v}_2 = \langle a_2, b_2 \rangle$. Let $A = \left[\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2 \end{smallmatrix}\right]$. Prove that the above conditions are also equivalent to $\det(A) = 1$.

(b) Let N be a submodule of $\mathbb{Z}^n$ of rank n, and let $A = (a_{ij})_{i=1}^n$ be the relations matrix for the generators of N with respect to the standard basis of $\mathbb{Z}^n$ (that is, the generators of N are:

$$y_1 = a_{11}e_1 + \dots a_{1n}e_n$$
$$\dots$$
$$y_n = a_{n1}e_1 + \dots a_{nn}e_n,$$

where $e_j = (0, \dots 0, 1, 0 \dots 0)$ with 1 in the jth place). Prove that

$$|\mathbb{Z}^n/N| = |\det(A)|.$$

Problem 8. In this problem, Parts (a) and (b) are needed only to explain what is going on in Part (c). If you solve Part (c), do not hand in (a) and (b). If you cannot solve (c), then you can write and hand in the solutions for (a) and (b).

(a) Sketch the ideal generated by $1 + 2i$ in $\mathbb{Z}[i]$. Find $|\mathbb{Z}[i]/(1 + 2i)|$.

(b) Sketch the ideal generated by 2 in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. Find $\left|\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]/(2)\right|$.

(c) Let $\mathcal{O}$ be a quadratic integer ring, and let $(a) \subset \mathcal{O}$ be a principal ideal in it. Recall that we have the notion of norm on quadratic integer rings. Prove that $|\mathcal{O}/(a)| = |N(a)|$, where $|N(a)|$ is the absolute value of the norm of a.
Hint: First show that $\mathcal{O}$ is a free rank 2 $\mathbb{Z}$-module. Then find a convenient set of generators of (a) as a $\mathbb{Z}$-submodule, and use the previous problem.