Correct solution for Problem 18 (b) from Webwork 7.

The problem had some randomized inputs, so I'll state a general version. Let us also rename the coordinates: I'll say E is x, and N is y (i.e., the x-axis points East and the y-axis points North). Then the problem was asking the following questions:

(a) Suppose the car is moving in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$, on the plane given by the graph of the linear function $z = c + ax + by$. (Here a, b, c were constants that were somewhat randomized, as well as the direction $\mathbf{u}$). Then, find the directional derivative $D_{\mathbf{u}} f$.

(b) Suppose the car is going in the same direction as in part (a), at the speed v. Then find how fast its altitude is changing with respect to time.

The answer to Part (a) was straightforward: you had to find the gradient of f (which is $\langle a, b \rangle$), and then $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = au_1 + bu_2$ (plug in the numbers from your version of the problem).

The answer to Part (b) that appears in Webwork solutions is: v times the answer from (a). At the first glance this is fine: if you go in the same direction but v times faster, then the change you experience will be v times greater. If the function $f(x, y)$ was measuring, say, temperature at the point (x, y) rather than your elevation over the point $(x, y, 0)$, and the car was moving in the horizontal plane at the speed v, this solution would have been fine. The problem is that the velocity of the car necessarily has 3 components, since the car is moving (as we are told in the problem) on this slanted plane.

Let $\mathbf{v}$ be the velocity of the car (a vector in $\mathbb{R}^3$), and let $\mathbf{v}'$ be the projection of the velocity of the car onto the xy-plane. Then it should be the magnitude of $\mathbf{v}'$, and *not* the magnitude of $\mathbf{v}$ that is used in the answer. In short, the answer to part (b) should be $|\mathbf{v}'|(au_1 + bu_2)$. But we are only given $\mathbf{v}$. Our task is to find the magnitude $|\mathbf{v}'|$. This is where we use the fact that as the car is moving in this plane in the direction of the vector $\mathbf{u}$, we know that for every unit of the distance in the xy-plane it is gaining $D_{\mathbf{u}} f$ units of elevation (exactly because the directional derivative measures the slope of the line along which the car is moving). Then we have: if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then $\mathbf{v}' = \langle v_1, v_2 \rangle$, and $v_3 = |\mathbf{v}'| D_{\mathbf{u}} f$. Then

$$|\mathbf{v}| = \sqrt{|\mathbf{v}'|^2 + |\mathbf{v}'|^2 (D_{\mathbf{u}} f)^2} = |\mathbf{v}'| \sqrt{1 + (D_{\mathbf{u}} f)^2}.$$

Finally, we get:

$$|\mathbf{v}'| = \frac{v}{\sqrt{1 + (D_{\mathbf{u}} f)^2}} = \frac{v}{\sqrt{1 + (au_1 + bu_2)^2}},$$

and the final answer is: the rate of change of altitude as the car moves with speed v in the direction of the unit vector $\mathbf{u}$ is

$$\frac{v}{\sqrt{1 + (au_1 + bu_2)^2}} (au_1 + bu_2).$$