Take-home mid-midterm. Due Wednesday November 18.

Problems:
(1) Let \( S \subset \mathbb{R} \) be the set
\[
S = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}, n \geq 1 \right\}.
\]
Find \( \partial S \) – the set of all boundary points of \( S \). Is the set \( S \) closed?

(2) Let \( S = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q}\} \). Find its boundary \( \partial S \).

(3) Using the \( \epsilon-\delta \) definition of a continuous function, prove that if \( f(x, y) \) is a continuous function on \( \mathbb{R}^2 \), and \( g(t) \) is a continuous function on \( \mathbb{R} \), then \( F(x, y) = g(f(x, y)) \) is continuous on \( \mathbb{R}^2 \).

(4) Using \( \epsilon-\delta \) definition, prove that
\[
\lim_{(x,y) \to (0,0)} \frac{xy^3}{x^6 + y^2} = 0.
\]

(5) For each real number \( r \), consider the function defined by
\[
f(x, y) = \begin{cases}
(x^2 + y^2)^r \sin \left( \frac{1}{x^2 + y^2} \right) & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases}
\]
Determine the set of all \( r \) such that at \( (0,0) \), the function \( f(x, y) \) is
(a) continuous
(b) differentiable
(c) continuously differentiable
You do not need to give \( \epsilon-\delta \) proofs, but need to justify your answers using the properties of limits, continuous, and differentiable functions.