Take-home mid-midterm. Due Wednesday November 18.

Problems:

(1) Let $S \subset \mathbb{R}$ be the set
$$S = \{\frac{1}{n} \mid n \in \mathbb{Z}, n \geq 1\}.$$
Find ∂S – the set of all boundary points of S. Is the set S closed?

(2) Let $S = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q}\}$. Find its boundary ∂S.

(3) Using the ϵ-δ definition of a continuous function, prove that if $f(x, y)$ is a continuous function on $\mathbb{R}^2$, and $g(t)$ is a continuous function on $\mathbb{R}$, then $F(x, y) = g(f(x, y))$ is continuous on $\mathbb{R}^2$.

(4) Using ϵ-δ definition, prove that
$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^6 + y^2} = 0.$$

(5) For each real number r, consider the function defined by
$$f(x, y) = \begin{cases} (x^2 + y^2)^r \sin\left(\frac{1}{x^2+y^2}\right) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$
Determine the set of all r such that at $(0, 0)$, the function $f(x, y)$ is
(a) continuous
(b) differentiable
(c) continuously differentiable
You do not need to give $\epsilon - \delta$ proofs, but need to justify your answers using the properties of limits, continuous, and differentiable functions.