Detailed solution to Problem 33, Section 10.1

We need to find (a) the interior points and (b) the boundary points of the set $S = \{(x, y) \in \mathbb{R}^2 | 0 < x^2 + y^2 < 1\}$.

First, some discussion (to set up the language – there were some problems with language in the homework): the *unit circle* is defined by the equation $x^2 + y^2 = 1$. Not that the circle itself, as a set, is very 'thin', it consists only of the points that are *precisely* at distance 1 from $(0, 0)$. Everything 'inside' it, that is, the points whose distance from $(0, 0)$ is less than 1, should not be called 'circle'; it is called *the disc* of radius 1 centred at $(0, 0)$. (In higher dimension, we would call this a *ball*, and its boundary – a *sphere*. In short, the disc has area, the ball has volume, but the circle does not have any area, and the sphere does not have any volume, or more precisely, they have area (respectively, volume) zero.).

With this terminology, **our set S is an open disc centred at $(0, 0)$ with the origin removed from it.**

Now we can discuss the solution of the problem.

Solution.

1. Interior points. First, let us prove that $\text{Int}(S) = S$, that is, that every point of the set S is an *interior point*. For that, we need to prove that for every point $s = (x_0, y_0) \in S$, there exists a disc which we will call $B_{s,r}$ of some radius r, centred at (x_0, y_0), that is contained in S entirely.[1] So, we are given (x_0, y_0) and we need to find the r that would work. If you draw a picture, you would see that any r that is smaller than the distance from s to the unit circle, and smaller than the distance from s to the origin, would work. But to be safe, let us take half that number: let $r = \min(\frac{1-d}{2}, \frac{d}{2})$, where $d = \sqrt{x_0^2 + y_0^2}$ is the distance from (x_0, y_0) to the origin. Note that $d > 0$, because the origin itself is *not* in S, so our r is positive. Now that we chose r, we want to prove that $B_{s,r} \subset S$. This means that we need to prove that for *any* point $p = (x, y) \in B_{s,r}$, we have $0 < x^2 + y^2 < 1$.

So, at this point, we are investigating the point $p = (x, y)$, and we know that the distance $d(p, s)$ from p to our point $s = (x_0, y_0)$ is less than r; and we want to prove that the distance from p to $(0, 0)$ is greater than 0 but less than 1. Let us denote this distance by $d(p)$ (so $d(p) = \sqrt{x^2 + y^2}$). We need to show

$$0 < d(p) < 1.$$

First, note that $d(p)$ cannot be zero: if it was, we would have that $p = (0, 0)$. But then the distance from p to s would be d, but because r is not greater than $d/2$, we know that the distance from p to s is not greater than $d/2$, which creates a contradiction. Thus, $d(p) > 0$. Now let us show that $d(p)$ is less than 1. By the triangle inequality, we have:

$$d(p) \leq d(p, s) + d.$$

Sine p is in the ball of radius r around s, we know that $d(p, s) < r$, and by our choice of r, we have $r < \frac{1-d}{2}$. Then

$$d(p, s) + d < \frac{1 - d}{2} + d = \frac{1 + d}{2}.$$

[1] Note that using the logic shorthand notation, we could write this sentence as:

$$\forall s = (x_0, y_0) \in S, \quad \exists r > 0 \text{ s.t. } B_{s,r} \subset S.$$

Since the point s is in the set S, we have $d < 1$, then $\frac{1+d}{2} < 1$. Thus we have proved that $d(p) < 1$. This completes the proof that the entire ball $B_{s,r}$ is contained in S, and thus we have proved that s is an interior point. Since s was an *arbitrary* point of S, we have completed the proof that $\mathrm{Int}(S) \supseteq S$, but since every interior point is automatically in S, we have $\mathrm{Int}(S) = S$.

2. Boundary points. The boundary of S is the set

$$\partial S = \{(x, y) \mid x^2 + y^2 = 1\} \cup \{(0, 0)\}.$$

This is saying that the set of boundary points of S is the unit circle and the origin. (Note that we already proved in class that the unit circle is the boundary of the disc of radius 1; but anyway, let us do it a bit differently here, using what we already proved about interior points).

Note a very important fact: if $\mathrm{Int}(S)$ is the set of interior points of S, and $\mathrm{Ext}(S)$ is the set of *exterior* points of S, then the boundary of S is the *complement of the union of these two sets*: symbolically, we can writ this as:

$$\partial S = (\mathrm{Int}(S) \cup \mathrm{Ext}(S))^c = \mathbb{R}^2 \setminus (\mathrm{Int}(S) \cup \mathrm{Ext}(S)).$$

You should be able to prove this from the definitions. Try it, and only then look at the footnote below that contains this proof. [2]

Now, we can prove that $\mathrm{Ext}(S) = \{(x, y) \mid x^2 + y^2 > 1\}$ exactly the same way as we did the proof for $\mathrm{Int}(S)$ (in fact, we also proved this in class). This leaves the boundary to be the answer above: the unit circle and the origin. (Of course, you could have checked this directly). Note that the origin does satisfy the definition of a boundary point for S: any small neighbourhood of it consists of the points of S, except the origin itself is in S^c !

3. Openness: The set S is open because every point is an interior point, as we proved above.

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[2]Proof of the fact that $\partial S = (\mathrm{Int}(S) \cup \mathrm{Ext}(S))^c$: in words, this says, that the point s is a boundary point precisely when it is neither an interior point nor an exterior point. Our statement is saying that the plane is divided into three non-overlapping sets: $\mathrm{Int}(S)$, $\mathrm{Ext}(S)$, and ∂S.

Here is a formal proof: By definition, $s \in \mathrm{Int}(S) \cup \mathrm{Ext}(S)$ if and only if

$$(\exists r > 0 \text{ s.t. } B_{s,r} \subset S) \text{ or } (\exists r > 0 \text{ s.t. } B_{s,r} \subset S^c).$$

Then the complement $(\mathrm{Int}(S) \cup \mathrm{Ext}(S)^c$ is defined by the *negation* of this condition. Now, let us negate the statement. We get:

$$(\forall r > 0 \;\; B_{s,r} \not\subset S) \text{ and } (\forall r > 0 \;\; B_{s,r} \not\subset S^c),$$

but this means precisely that for *any* ball $B_{s,r}$ centred at s, this ball contains both points of S and points of S^c, which is precisely the definition of the boundary point!