~~Rip~~ Review Session MATH 226

Integrals

3^d: spherical

Topics:
- * vectors, planes, line ...
- * quadratic surfaces
- * limits, continuity, definition of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$
- * gradients
- * chain rule
- * Implicit, Jacobians,
- * critical pts, Hessian
- * Lagrange
- * Integration: change of variables

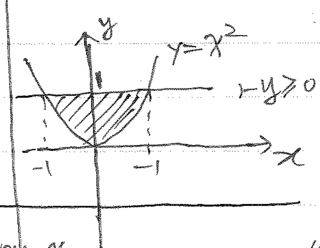
eg 1 eg: 2003 Exam Problem 7.

R - solid, $y \geq x^2$, $0 \leq z \leq 1-y$

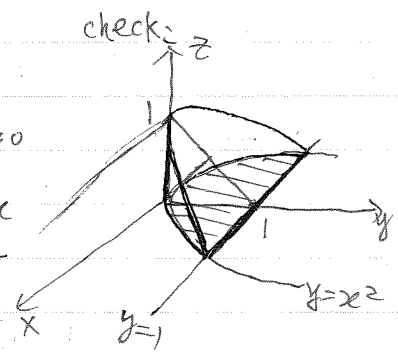
① $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)\, dz\, dy\, dx$

② $\int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-z} f(x,y,z)\, dy\, dz\, dx$

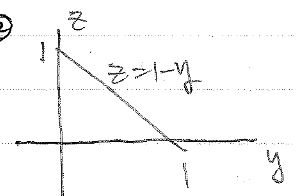
① sketch:



check:



②



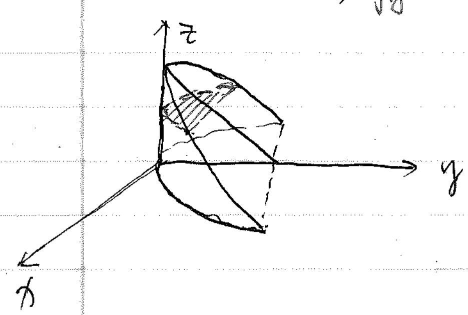
find bound for z given x

Know: $z \geq 0$, $z \leq 1-y$, but $y \geq x^2$

$\Downarrow$ $y \leq 1-z$

$y \geq x^2 \Rightarrow 1-y \leq 1-x^2 \Rightarrow z \leq 1-y \leq 1-x^2 \Rightarrow z \leq 1-x^2$

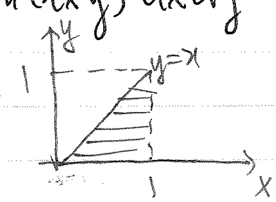
③ $\int_{0}^{1}\int_{0}^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\, dx\, dy\, dz$



2. eg: integral $\int_0^1\int_y^1 x^2 \sin(\pi x y)\, dx\, dy$

solution: switch order

$\int_0^1\int_0^x x^2 \sin(\pi x y)\, dy\, dx$



eg 3: $\int_{x=1}^{2}\int_{y=x}^{2}\int_{z=\ln x}^{\ln y}$ dzdydx

$\int\int\int$ dxdzdy $\qquad \int\int\int$ dydxdz.

know:

$$\boxed{\begin{array}{c} 1 \le x \le 2 \\ x \le y \le 2 \\ \ln x \le z \le \ln y \end{array}}$$

$$\int_1^2 \int_0^{\ln y} \int \quad dx\,dz\,dy$$

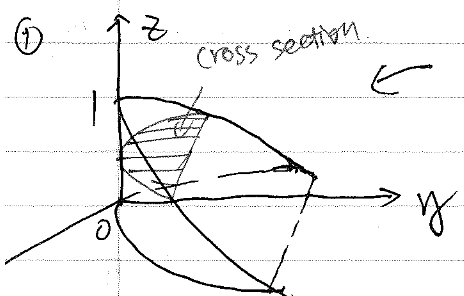
$$1 \le y \le 2$$

$$0 = \cancel{\ln 1} \le \ln x \le z \le \ln y$$

$$\left.\begin{array}{l} x \le y \\ x \le e^z \le y \end{array}\right\} \Rightarrow \text{compete, must satisfy the two equation.}$$

$$1 \le x \le e^z$$

Projections VS Cross-Sections.

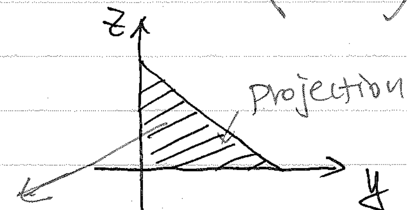
① 

$$\leftarrow \int_0^1 \left(\int \int \quad dx\,dy \right) dz$$

Cross section fixed z

(integral z for individual slice)

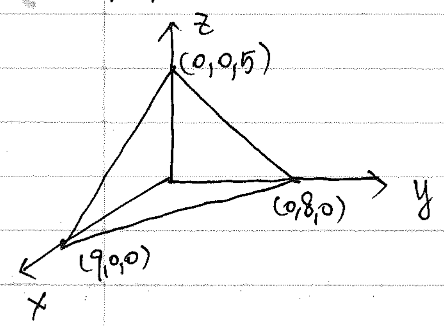
② $\int \int \left(\int \quad dx \right) dy\, dz$



integral the whole area

for individual line

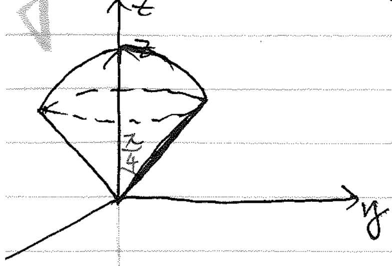
$$\int \int \int xy\, dV$$



(0, 0, 0) (9, 0, 0) (0, 8, 0) (0, 0, 5)

Cross product $\rightarrow$ equation of the plane

9: Sperical. Cylindrical coordinates: $\sqrt{x^2+y^2} \le z \le 12 - x^2 - y^2$



$$\int_0^{2\pi} \int_0^3 \int_r^{12-r^2} dz\; r\,dr\,d\theta$$

$z = c\sqrt{x^2+y^2}$

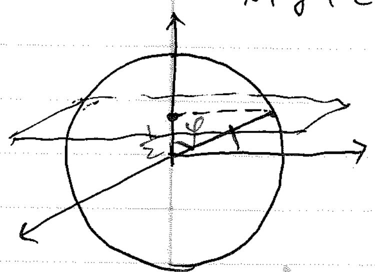
$\tan\varphi = \frac{1}{c}$

$$\sqrt{x^2+y^2} \le 12 - x^2 - y^2 \Rightarrow r \le 12 - r^2$$

$r = 12 - r^2 \qquad 0 \le r \le 3$

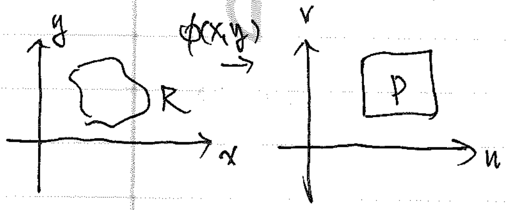
$$\int_0 \int \int z(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$x^2+y^2+z^2 \leq 1 \quad z \geq \frac{1}{2} \qquad z = \rho\cos\varphi = \frac{1}{2}$



$$\int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_{\frac{1}{2\cos\varphi}}^{1} \rho\cos\varphi\,(\rho^2)^{-\frac{3}{2}} \rho^2\sin\varphi \; d\rho\, d\varphi\, d\theta$$

eg: change of variables for double integral.



$\phi(x,y)$

$u = u(x,y)$

$v = v(x,y)$

1> check if one to one

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

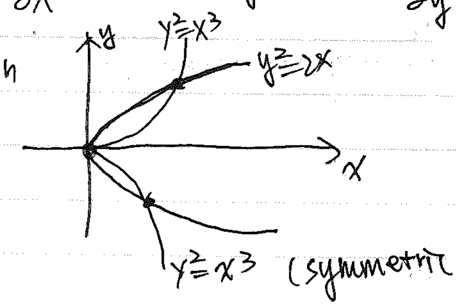
2> $\iint_D f(u,v)\, du\, dv = \iint_R f(\phi(x,y))\, |\text{Jacobian}|\, dx\, dy$

eg: critical points, Hessian

$f(x,y) = x^4 + y^4 - 4xy^2$

$\frac{\partial f}{\partial x} = 4x^3 - 4y^2 = 0 \qquad \frac{\partial f}{\partial y} = 4y^3 - 8xy = 0 \qquad \begin{cases} x^3 = y^2 \\ y^2 = 2x \end{cases}$

graph



critical points $(0,0)$ $(\sqrt{2}, 2^{\frac{3}{4}})$ $(\sqrt{2}, -2^{\frac{3}{4}})$

$f_{xx} = 12x^2$

$f_{xy} = -8y$

$f_{yy} = 12y^2 - 8x$

eg: Implicit 2003 Exam # 3

$f(\bar{0}) = 0$ ($\to \mathbb{R}^4$) $\quad g(\bar{0}) = 0 \quad \nabla f(\bar{0}) = \langle 2, 3, 0, -1\rangle \quad \nabla g(0) = \langle 2, 1, -1, -2\rangle$

(1) $\left(\frac{\partial u}{\partial x}\right)_y \quad \left(\frac{\partial u}{\partial x}\right)_v \quad \begin{cases} f(u, v, x, y) = 0 \\ g(u, v, x, y) = 0 \end{cases}$

$$\begin{bmatrix} \partial u/\partial x \\ \partial v/\partial x \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial g}{\partial x} \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(2) $DF(\bar{0}) \quad F: \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x, y) = \begin{bmatrix} f(u(x,y), v(x,y), x, y) \\ g(u(x,y), v(x,y), x, y) \end{bmatrix}$$

$$\underset{\mathbb{R}^2}{(x,y)} \xrightarrow{G} \underset{\mathbb{R}^4}{(u(x,y), v(x,y), x, y)} \xrightarrow{H} \begin{bmatrix} f(u(x,y), v(x,y), x, y) \\ g(u(x,y), v(x,y), x, y) \end{bmatrix}$$

$u(x,y) = e^x \cos y + xe^y + y - 1$

$v(x,y) = e^y(1 - \cos x) + \sin 3x$

$DF = DH \cdot DG$ (Jacobian Matrix)

$F(x, y) = H(G(x,y))$

when $(x, y) = (0, 0)$ $\quad G(0,0) = (u(0,0), v(0,0), 0, 0) = (0, 0, 0, 0)$

$$DF|_{(0,0)} = DH|_{\substack{G(0,0) \\ (0,0,0,0)}} \cdot DG|_{(0,0)}$$

$$= \begin{bmatrix} 2 & 3 & 0 & -1 \\ 2 & 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

eg: Fall 2007. # 6. Limits, continuity, differentiability.

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

1) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\Big|_{(0,0)}$

$$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \lim_{h \to 0} \frac{f(h, 0) - f(0,0)}{h} = \frac{\frac{h^2 \cdot 0}{h^2 + 0}}{h} = 0$$

$$\frac{\partial f}{\partial y}\Big|_{(0,0)} = \lim_{h \to 0} \frac{f(0, h) - f(0,0)}{h} = \frac{0}{h} = 0$$

2) differentiablity

$$\lim_{(h,k) \to (0,0)} \frac{f(h, k) - f(0,0) - h \cdot 0 - k \cdot 0}{\sqrt{h^2 + k^2}} = \lim_{(h,k) \to (0,0)} \frac{h^2 k}{(h^2 + k^2)^{\frac{3}{2}}}$$

$h = mk \quad \therefore \lim_{(h,k) \to (0,0)} \frac{m^2 k^3}{(m^2 k^2 + k^2)^{\frac{3}{2}}}$

3) $D_{\bar{u}} f(0,0) \quad \bar{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle \quad \nabla f(0,0) = \langle 0, 0 \rangle$

$$\| \lim_{h \to 0} \frac{f((0,0) + h\bar{u}) - 0}{h} = \lim_{h \to 0} \frac{f\left(\frac{h}{\sqrt{5}}, \frac{2h}{\sqrt{5}}\right) - 0}{h}$$

eg: 2001, #6

$f(x,y,z) = x+y^2z$ D: $2x^2+y^2+z^2 \le 1$ has abs min and max

1) inside critical points ~~i~~

2) boundary ← Lagrange.

1) critical points : $\frac{\partial f}{\partial x}=1=0$ $\frac{\partial f}{\partial y}=2yz=0$ $\frac{\partial f}{\partial z}=y^2=0$

impossible ⇒ no critical points

2) Lagrange : $x+y^2z+\lambda(2x^2+y^2+z^2-1)=0$

$$\begin{cases} 1+4\lambda x=0 \\ 2yz+2\lambda y=0 \\ y^2+2\lambda z=0 \\ 2x^2+y^2+z^2-1=0. \end{cases}$$

eg: $z+x^2+z^4+y^2$ on R^3, abs max/min ?

critical points : $\begin{cases} 2x=0 \\ 2y=0 \\ 4z^3+1=0 \end{cases}$ $(x,y,z)=(0,0,\sqrt[3]{-\frac{1}{4}})$

Implicit function

u, v, x, y $\quad\quad f, g : R^4 \to R$

$f(\bar{0}) = 0 = g(\bar{0})$, $\bar{0} \in R^4$

$\nabla f(0) = \langle 2, 3, 0, -1 \rangle$

$\nabla g(0) = \langle 2, 1, -1, -2 \rangle$

u, v dep

x, y indep

$$\begin{cases} f(u, v, x, y) = 0 \\ g(u, v, x, y) = 0 \end{cases} \quad \text{Compute } \left(\frac{\partial u}{\partial x}\right)_y \quad \left.\frac{\partial u}{\partial x}\right|_v$$

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial x} = 0$$

$$\frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial g}{\partial x} = 0$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix}^{-1} \cdot \begin{bmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial g}{\partial x} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ -(-1) \end{bmatrix}$$

$*$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \frac{1}{\det}$

$\det = ad - bc$

...

$DF(0)$ $\quad F: \mathbb{R}^2 \to \mathbb{R}^2 \quad F(x,y) = \begin{bmatrix} f(u(x,y), v(x,y), x, y) \\ g(u(x,y), v(x,y), x, y) \end{bmatrix}$

$u(x,y) = e^x \cos y + x \cdot e^y + y - 1$

$v(x,y) = e^y(1 - \cos x) + \sin 3x$

$$(x,y) \xrightarrow{G} (u(x,y), v(x,y), x, y)$$

$\mathbb{R}^2 \qquad \mathbb{R}^4 \downarrow H$

$f \ldots$

$g \ldots$

$\downarrow$

$\mathbb{R}^2$

$\cancel{DF = DG}$

$F(x,y) = H(G(x,y))$

$DF = DH \cdot DG$

when $(x,y) = (0,0)$

$G(0,0) = (0, 0, 0, 0)$

$$DF\Big|_{(0,0)} = DH\Big|_{(0,0,0,0)} \cdot DG\Big|_{(0,0,0,0)}$$

$$\begin{bmatrix} 2 & 3 & 0 & -1 \\ 2 & 1 & -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 \end{cases}$$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - 0 - 0}{\sqrt{h^2+k^2}} = \lim \frac{h^2k}{(h^2+k^2)\sqrt{h^2+k^2}}$$ powers

let
$h = mk$

$$= \frac{m^2k^2 \cdot k}{(m^2k^2+k^2)^{\frac{3}{2}}} = \frac{m^2}{(m^2+1)^{\frac{3}{2}}}$$ doesn't exist not Diff....

works only differentiable!

$$D_{\vec{u}} f|_{(0,0)} \neq \nabla f|_{(0,0)} \cdot \vec{u}$$

Def $= \lim_{h\to 0} \dfrac{f(\frac{h}{\sqrt5}, \frac{2h}{\sqrt5}) - 0}{h}$

Lagrange. Continuous & Closed Domain

$f(x,y,z) = x + y^2 z \qquad D: 2x^2 + y^2 + z^2 \le 1$

has abs min / max

① critical pts

② lagrange → boundaries

1) $\dfrac{\partial f}{\partial x} = 1 \qquad \dfrac{\partial f}{\partial y} = 2yz \qquad \dfrac{\partial f}{\partial z} = y^2$

No critical point! $\qquad \nabla g = \langle 4x, 2y, 2z \rangle$

$$\begin{cases} 1 = \lambda 4x \\ 2yz = \lambda \cdot 2y \\ y^2 = \lambda \cdot 2z \\ 2x^2+y^2+z^2 = 1 \end{cases} \Rightarrow 6 \text{ points}$$