*This midterm has **4 questions** on **7 pages**, for a total of 50 points.*

Duration: 50 minutes

- **you need to justify all your answers.**
- Continue on the back of the previous page if you run out of space.
- One page formula sheet is allowed. No other documents, and no electronic devices of any kind (including calculators, cell phones, etc.) are allowed.
- Unless a problem states otherwise, you **do not** have to simplify algebraic expressions to the shortest possible form, and do not have to evaluate long numerical expressions or invert any matrices.

Full Name (including all middle names): ______________________

Student-No: ______________________

Signature: ______________________

Question:	1	2	3	4	Total
Points:	15	7	16	12	50
Score:					

1. Consider the surface defined by the equation $y^2 z = x^3 + 2xz^2 + z^3$.

4 marks (a) Find $\frac{\partial x}{\partial z}$ at the point $(1, 2, 1)$.

4 marks (b) Find the equation of the tangent plane to this surface at the point $(1, 2, 1)$.

3 marks (c) Find a condition on (x_0, y_0, z_0) that guarantees that the point (x_0, y_0, z_0) on this surface has a neighbourhood on which x can be represented as a differentiable function of (y, z).

4 marks (d) Find the second order partial derivative $\frac{\partial^2 x}{\partial z^2}$ at $(1, 2, 1)$.

Solution: (a). Let $F(x, y, z) = x^3 + 2xz^2 + z^3 - y^2 z$. We compute: $F_x = 3x^2 + 2z^2$; $F_z = 3z^2 + 4xz - y^2$. Then

$$\frac{\partial x}{\partial z} = -\frac{F_z}{F_x} = -\frac{z^2 + 4xz - y^2}{3x^2 + 2z^2}.$$

Evaluating at $(1, 2, 1)$, get $-3/5$.

(b). Let us also compute $F_y = -2yz$. We get: $F_x(1, 2, 1) = 5$, $F_y(1, 2, 1) = -4$, $F_z(1, 2, 1) = 3$. Then $\nabla F|_{(1,2,1)} = \langle 5, -4, 3\rangle$. Our surface is the level surface with the equation $F(x, y, z) = 0$. Then $\nabla F|_{(1,2,1)}$ is normal to its tangent plane at $(1, 2, 1)$. Then the equation of the tangent plane is

$$5(x - 1) - 4(y - 2) + 3(z - 1) = 0.$$

Note that you could also have used the implicit derivative found in (a) to get this equation (thinking of x as a function of y, z, and using the partial derivative you found, and similarly computed $\frac{\partial x}{\partial y}$). Then you would obtain the equation in the form:

$$x = 1 + \frac{4}{5}(y - 2) - \frac{3}{5}(z - 1),$$

which is an equivalent equation.

(c). The condition is that $F_x \neq 0$ (this is the denominator when you commute the implicit derivative as in (a)). By Implicit Function theorem (in our case, with just one equation and one dependent variable), x is a differentiable implicit function of (y, z) in a neighbourhood of a point (x_0, y_0, z_0) if $\frac{\partial F}{\partial x}|_{(x_0,y_0,z_0)} \neq 0$, which means, $3x_0^2 + 2z_0^2 \neq 0$, which is equivalent to saying that both x_0 and z_0 are not simultaneously zero.

(d). Differentiate the expression we got in (a) with respect to z again, but remembering that every time we see x, we need to think of it as a function of y and z. We get (using quotient rule, and chain rule, etc. every time we see x):

$$\frac{\partial^2 x}{\partial^2 z} = \frac{\partial}{\partial z}\left(-\frac{z^2 + 4xz - y^2}{3x^2 + 2z^2}\right)$$
$$= -\frac{(3x^2 + 2z^2)(2z + 4z\frac{\partial x}{\partial z} + 4x) - (z^2 + 4xz - y^2)(6x\frac{\partial x}{\partial z} + 4z)}{(3x^2 + 2z^2)^2}.$$

It remains to recall that we know the value of $\frac{\partial x}{\partial z}$ at $(1, 2, 1)$ from (a), plug it in and evaluate. We get:

$$\frac{\partial^2 x}{\partial^2 z}|_{(1,2,1)} = -\frac{5(2 - 4 \cdot \frac{3}{5} + 4) - (-6 \cdot \frac{3}{5} + 4)}{25}.$$

7 marks

2. Suppose you know the following facts about air temperature over Salt Lake City at 9am on July 1:

- the temperature decreases by 1° (in degrees C) per 100 metres increase in altitude;
- the temperature does not change along the y-axis (which points North);
- the temperature decreases by 0.2° for every 100 metres one moves in the East direction (and the x-axis points East);
- the temperature increases by 5° per hour.

A hot air balloon is travelling over Salt Lake City at that moment (9am on July 1), with velocity $\mathbf{v} = \langle 3, 2, 0.5 \rangle$ (in km/hour). What is the rate of change of temperature that the passengers are experiencing?

Solution: Let $h(t) = T(x(t), y(t), z(t), t)$ be the temperature that the passengers feel, as a function of time. By Chain Rule, we get:

$$\frac{dh}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} + \frac{\partial T}{\partial t}.$$

We know that at t equal to 9am on July 1, the values are:

$$\frac{\partial T}{\partial x} = -0.2, \quad \frac{\partial T}{\partial y} = 0, \quad \frac{\partial T}{\partial z} = -1,$$

measured in degrees per 100m, and $\frac{\partial T}{\partial t} = 5$, measured in degrees per hour. If we want to measure the rate of change of temperature in degrees per hour, we need to measure velocities in "hundreds of metres" per hour. Note that the instantaneous velocity of the balloon at any time t is the vector of derivatives: $\mathbf{v} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle = \langle 30, 20, 5 \rangle$, where the units are 100m/h. Plugging it all in, we get:

$$\frac{dh}{dt} = -0.2 \cdot 30 + 0 + (-1) \cdot 5 + +5 = -6 - 5 + 5 = -6 \text{ degrees per hour}$$

4 marks

3. (a) Is the function $h(x, y)$

$$h(x,y) = \begin{cases} \frac{xy^3}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

continuous at $(0, 0)$? (You do not have to use ϵ and δ, but you have to provide a careful explanation why or why not.)

Solution: Consider the limit along the lines $y = kx$ for different values of k. We get:

$$h(x, kx) = \frac{k^3x^4}{(x^2 + k^2x^2)^2} = \frac{k^3}{(1 + k^2)^2},$$

which is constant; thus, the limit of $h(x, kx)$ as $x \to 0$ is $\frac{k^3}{(1+k^2)^2}$, which depends on k. Therefore, the limit $\lim_{(x,y)\to(0,0)} h(x, y)$ does not exist, so this function is *not* continuous at the origin.

Note: it was enough, as many have done, to consider any k, e.g. $k = 1$, and show that this limit is not 0.

3 marks

(b) Let

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Find $\frac{\partial f}{\partial x}|_{(0,0)}$.

Solution: The key point is that we cannot just differentiate the formula defining $f(x, y)$ at points $(x, y) \neq (0, 0)$, because that would only give us the values of $\frac{\partial f}{\partial x}$ at these other points, and if we take the limit as $(x, y) \to (0, 0)$, we will get (if the limit exists) $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}$, but *not* the value $\frac{\partial f}{\partial x}|_{(0,0)}$. Thus, we simply need to use the definition of a partial derivative at a point:

$$\frac{\partial f}{\partial x}|_{(0,0)} = \lim_{h\to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{f(h, 0)}{h} = 0,$$

since $f(h, 0) = 0$ for all h. Answer:

$$\frac{\partial f}{\partial x}|_{(0,0)} = 0.$$

2 marks

(c) For the function $f(x, y)$ from Part (b), is the function $\frac{\partial f}{\partial x}$ continuous at $(0, 0)$?

Solution: This is where we need to differentiate the formula defining $f(x, y)$ at points $(x, y) \neq (0, 0)$, and then determine if the limit as $(x, y) \to (0, 0)$ exists and equals 0, which is the value of our function $\frac{\partial f}{\partial x}$ at $(0, 0)$ by the previous question.

Differentiating, we get that in fact at $(x, y) \neq (0, 0)$, $\frac{\partial f}{\partial x}(x, y) = 2h(x, y)$, where $h(x, y)$ is the function from part (a). Since we know it does not have a limit at $(0, 0)$, we obtain that $\frac{\partial f}{\partial x}(x, y)$ is discontinuous at $(0, 0)$.

4 marks

(d) Using the function $f(x, y)$ from Part (b), find the directional derivative $D_{\mathbf{u}}f$ in the direction of the vector $\mathbf{u} = \langle 1, k\rangle$, where k is a constant.

Solution: Note that the formula $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$ is guaranteed to work only when f is continuously differentiable at the given point. We know from part (c) that this is not the case, so we cannot use this formula, and have to use the definition of the directional derivative. First, we need to make our $\mathbf{u}$ into a unit vector: consider

$$\mathbf{u_1} = \frac{1}{|\mathbf{u}|}\mathbf{u} = \langle \frac{1}{\sqrt{k^2+1}}, \frac{k}{\sqrt{k^2+1}}\rangle.$$

Then by definition of the directional derivative,

$$D_{\mathbf{u}}f = \lim_{t\to 0} \frac{f(\frac{t}{\sqrt{1+k^2}}, \frac{tk}{\sqrt{1+k^2}}) - f(0,0)}{t}$$
$$= \lim_{t\to 0} \frac{t^3k}{t\cdot t^2(1+k^2)\sqrt{1+k^2}} = \frac{k}{(1+k^2)\sqrt{1+k^2}}.$$

3 marks

(e) Is the function $f(x, y)$ differentiable at $(0, 0)$?

Solution: The answer is "No". First, the same way as in (b), we compute that $\frac{\partial f}{\partial y}|_{(0,0)} = 0$. From here, there are two ways to prove that $f(x, y)$ is not differentiable. One is to write the limit from the definition of 'differentiable' (note that by (b), and the above calculation of $\frac{\partial f}{\partial y}|_{(0,0)}$, we get that the linearization of $f(x, y)$ at the point $(0, 0)$ is identically zero. Then the limit we need to investigate is:

$$\lim_{h,k\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}},$$

and the function is differentiable at $(0, 0)$ if and only if this limit exists and equals zero. One can see that the limit does not exist plugging in $y = cx$ for different values of c, as in part (a).

The faster solution is to say that if $f(x, y)$ were differentiable at $(0, 0)$, then the chain rule would have applied, and we would have had the formula

$$D_{\mathbf{u}}f(0,0) = \nabla f|_{(0,0)} \cdot \mathbf{u},$$

for any unit vector $\mathbf{u}$. We established in (b) and above in this solution that $\nabla f|_{(0,0)} = \mathbf{0}$, which would then have forced all the directional derivatives at $(0, 0)$ to be 0. On the other hand, we have just computed in (d) that there are non-zero directional derivatives, which leads to a contradiction, proving that $f(x, y)$ is not differentiable at $(0, 0)$.

4. Consider the transformation $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(s,t) = (e^s t, ts^2)$, and the transformation $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(u,v) = (u^2+v^2, u^2-v^2, 2uv)$. Let h be the composition: $h(s,t) = f(g(s,t)) = (h_1(s,t), h_2(s,t), h_3(s,t))$ (that is, h_1, h_2, h_3 are the individual coordinate functions of the transformation h).

4 marks

(a) Find $\frac{\partial h_1}{\partial s}|_{(1,3)}$.

Solution: We have $h_1(s,t) = u(s,t)^2 + v(s,t)^2$. One possible solution was to plug in the formulas for u and v and differentiate: $h_1(s,t) = (e^s t)^2 + (ts^2)^2$, then differentiate this formula with respect to s. This was not intended (I did not notice how easy it is to plug these formulas in), but it received full marks if done correctly.

The intended solution: by Chain rule,

$$\frac{\partial h_1}{\partial s} = \frac{\partial h_1}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial h_1}{\partial v}\frac{\partial v}{\partial s} = 2ue^s t + 2v \cdot 2st.$$

Evaluate at $(s,t) = (1,3)$: then $(u,v) = (3e, 3)$, and we get:

$$\frac{\partial h_1}{\partial s}|_{(1,3)} = 6e \cdot 3e + 2 \cdot 3 \cdot 6 = 18e^2 + 36.$$

4 marks

(b) Find the differential Dh of h.

Solution: The differential is the linear transformation determined by the Jacobian matrix of h. This Jacobian matrix is (the first equality is the matrix form of Chain rule):

$$\begin{bmatrix} \frac{\partial h_1}{\partial s} & \frac{\partial h_1}{\partial t} \\ \frac{\partial h_2}{\partial s} & \frac{\partial h_2}{\partial t} \\ \frac{\partial h_3}{\partial s} & \frac{\partial h_3}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \\ \frac{\partial h_3}{\partial u} & \frac{\partial h_3}{\partial v} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} = \begin{bmatrix} 2u & 2v \\ 2u & -2v \\ 2v & 2u \end{bmatrix} \cdot \begin{bmatrix} te^s & e^s \\ 2ts & s^2 \end{bmatrix}$$

4 marks

(c) Consider just the transformation $f(u,v)$. Its image is a surface in $\mathbb{R}^3$; let us call this surface S. Suppose Professor Anstee drew the unit circle on the (u,v)-plane before applying the transformation f to it. Then f transforms this circle to some curve on the surface S. Find the equation of the tangent line to this curve at the point $(1,-1,0)$.

Solution: First, note that the point $(1,-1,0)$ is the image, under our transformation, of the points $(0,\pm 1)$: $(1,-1,0) = f(0,1) = f(0,-1)$. Indeed, if we solve the equations $u^2+v^2 = 1$, $u^2-v^2 = -1$, $2uv = 0$, we get $u = 0, v = \pm 1$.

From here, there were two solutions: one is to note that the circle $u^2+v^2 = 1$ that Prof. Anstee drew is transformed by f to the unit circle, call it C, lying in the vertical plane $x = 1$ (in fact, you can check that the coordinates $(x,y,z) = (u^2+v^2, u^2-v^2, 2uv)$ satisfy $x^2 = y^2+z^2$, so the transformation f takes the uv-plane to the cone around the x-axis). Then you can just see that the tangent line to the circle C at the point $(1,-1,0)$ is parallel to the z-axis.

The intended solution was the following. Imagine that Prof. Anstee also drew a unit tangent vector to his circle at the point $(0,1)$. That would be the vector $\langle 1,0\rangle$ on the uv-plane. We know that the differential at the point $(u,v) = (0,1)$ of the transformation f is a linear transformation, so it would take this tangent vector to a vector in $\mathbb{R}^3$. Since the differential provides the best linear approximation to f at this point, it has to take the vector that was tangent to Prof. Anstee's circle to a vector that is tangent to its image. Thus, all we need to do in order to find the direction vector of the tangent line to the curve C (the image of Prof. Astee's circle), is to find the image of the vector $\langle 1,0\rangle$ under Df. We have already computed the matrix of Df (it is the Jacobian matrix of f):

$$Df|_{(u,v)} = \begin{bmatrix} 2u & 2v \\ 2u & -2v \\ 2v & 2u \end{bmatrix}.$$

We evaluate it at $(0,1)$, and get

$$Df|_{(0,1)} = \begin{bmatrix} 0 & 2 \\ 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

Then the direction vector of the tangent line to C is

$$\mathbf{v} = Df|_{(0,1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Then the tangent line to C at $(1,-1,0)$ has the parametric equation

$$x = 1, y = -1, z = t.$$