Writing proofs: a note about language and quantifiers

First, some notes about language and notation. If you are familiar with it, skip to the second page.

Set notation. We will often use the following kind of phrases:

> '*Let s be an element of the set S ...* ' (for example, '*Let a point (x, y) belong to the given set S...*').

The standard way to write this symbolically is:

> 'Let $s \in S$' (in the example, 'let $(x, y) \in S$').

The symbol $\in$ is always used with an *element* on the left and a *set* on the right. You can also write '$S \ni s$', which has the same meaning – the set S *contains* the element s.

When you want to say that a set A is contained in another set B, use $A \subseteq B$ or $B \supseteq A$ (again, the meaning is the same here: A is contained in B, or, equivalently, B contains A). You can use the symbols $\subseteq$, $\subset$, $\supseteq$, $\supset$ when you have *sets* on both sides.

Quantifiers. We will use quantifiers '$\forall$', which is called *universal quantifier* and '$\exists$', which is called *existential quantifier*. They will always be used to say something about elements of some set:

> '$\forall x \in A$' means '*For all elements x of the set A ...*'
> '$\exists x \in A$' means '*Exists an element x in the set A such that ...*'.

Note that we always need to use the words 'such that' (often abbreviated 's.t.') when we use the existential quantifier or the word 'exists', in order to specify what kind of element we want to exist.

For example, '$\exists x \in \mathbb{R}, x^2 - 3x + 2 = 0$' is read as 'there exists a real number x *such that* $x^2 - 3x + 2$ is zero'.

Implications and logical equivalences. When making proofs, we will often use *implication* and *equivalence* of statements. These notions have formal definition in the field of formal logic.

A sentence is called a *statement* if it is either true or false (though we do not need to know right away which it is). Consider, for example, the sentences: 'There exists a real number x such that $x^6 + 17x^5 + 30x^3 + x + 1 = 0$'. It is not at all clear right away whether it is true or false, but it is a *statement*, because only one of these possibilities holds: either this polynomial has a root or it doesn't. 'It will rain tomorrow' is also a statement, because again even though we do not know what will happen, either it will rain or it won't; there is no third possibility, and these two possibilities are mutually exclusive. In contrast, the sentence 'don't listen to your parents' is not a statement, because it cannot be assigned a true/false value (regardless of whether you agree or not with what it says). [1] You can **only use sentences that are statements** when writing proofs.

We say that a statement A *implies* the statement B if we know that if A is true, B has to be true (if A is false, B can be anything). (This is not the formal definition of implication, but it will work for this course).

[1] If you want to practice finding non-statements, politician's speeches are generally a good source. And this sentence itself is a non-statement.

For example, the statement about a *real* number x

‘$\exists y \in \mathbb{R}$ s.t. $y^2 = x$’

implies the statement ‘$x \geq 0$’. (try writing it down in words without any symbols and see if you agree).

The statements A and B are called *equivalent* if they imply each other. In this case we say A holds *if and only if* B holds.

For example, the statements “$x > 2$” and “x is positive and $x^2 > 4$” are equivalent. (Exercise: if you remove “x is positive”, will they stay equivalent? Will one of them imply the other still?).

Notation: ‘$A \Rightarrow B$’ means ‘A implies B’; ‘$A \Leftrightarrow B$’ means ‘A is equivalent to B’.

Negation. A statement B is called the *negation* of the statement A if:

B is true if and only if A is false.

This takes some practice: The negation of ‘He has black hair’ is ‘He does not have black hair’ (not ‘He is blond’).

Very important example.
Let the statement A say ‘All my friends have black hair’.
Let the statement B say ‘I have a friend who does not have black hair’.
Let the statement C say ‘All my friends do not have black hair’.
Which is the correct negation of the statement A? Is it B or C?

Answer: It is B. Think about it: the negation is such a statement that has to be true in **all** cases when A is false. There should be no third possibility: either A is true, or B is true; they cannot be both true at the same time and cannot be both false at the same time. This works for A and B. But for A and C, this doesn't work: I might have one friend with black hair, and one friend with red hair, for example; in this situation neither A nor C is true, so C cannot be the negation of A.

Negating statements with quantifiers. There is a simple rule for negating statements. Formal logic *proves* that this rule works. You should at least convince yourself with examples that it does work, and practice making negations (make sure everything you say makes sense to you!).

To negate a complex logical statement, you do the following:

- If there are quantifiers, all quantifiers switch: replace $\forall$ with $\exists$ and $\exists$ with $\forall$;
- Replace every ‘and’ with ‘or’ and every ‘or’ with ‘and’.
- negate (that is, put a ‘not’ in front of) every simple statement (by simple, I mean the ones without any more quantifiers, ‘and’s and ‘or’s inside).

Example: Negate the statement ‘all dogs either wag or growl and bite when they first meet you’.

Answer: ‘There exists a dog that does not wag and does not growl or bite when it first meets you’.

More than one quantifier: things get more complicated when the statement contains more than one quantifier. In this course, we will often need statements with two or three quantifiers. If you are uncomfortable with them, please read the separate handout on double quantifiers before continuing.

Example: Negate the statement

(1) $\exists x \in \mathbb{R},\ \text{s.t. } \forall y \in [x-1, x+1],\ y \geq -x.$

Answer: The negation is:

$$\forall x \in \mathbb{R}, \exists y \in [x-1, x+1], s.t.\ y < -x.$$

Discussion: Let us see what the statement (1) says, and try to determine if it is true. The statement says that there exists a real number x, such that if you take *any* number y from the interval $[x-1, x+1]$, this number y will satisfy $y \geq -x$. Do you think this statement is true or false?

Now suppose x is fixed. Consider the statement *about* x:

(2) $\forall y \in [x-1, x+1],\ y \geq -x.$

Plot x on the real line, plot the interval $[x-1, x+1]$, and find a simple condition on x (in the form of inequalities) which would be *equivalent* to this statement. Then write its negation and do the same for the negation. Note the relationship between the set of elements x for which (2) is true and the set of elements x for which its negation is true.

More exercises.

(1) Let S be the set of all students in our class, and for every student $s \in S$, denote by $F(s)$ the set of all friends of s. For every person s, let $N(s)$ be the set of neighbours of the person s (let us say that people are neighbours if they live in the same building). If s is a person, let $f(s)$ be the age of the person s. Using this notation, rewrite the statement

  'Every student in the class has a friend such that all neighbours of this friend are younger than 50'.

(use the quantifiers and the symbol $\in$). Then write down the negation of this statement.

Do you have a friend who lives in a dorm on campus? If yes, can you conclude whether this statement is true about our class?

Suppose you lived in a dorm, and you were friends with *everybody* in the class. Could you then conclude whether this statement about our class is true?

Now think of all your friends. Suppose you somehow knew that every friend of yours has a neighbour who is at least 50 years old. Could you then conclude if this statement about our class is true or false?

(2) Let us say that two trees are 'neighbours' if the distance between them is less than 15 meters. Negate the statement:

  'There exists a tree in Stanley park such that all the neighbouring trees are at least as tall as this tree.'

Write down the negation of this statement. Do you think this statement is true? Why or why not? Do you have to go to Stanley Park and look at every tree and its neighbours in order to find out? If not, why not? *Big Hint: There are finitely many trees in Stanley Park, and among them, there has to be the shortest one.*

(3) Write the negation of the statement:

  For every number $h < 100$ there exists a tree in Stanley Park that has a neighbour that is at least h meters tall.

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