Limits of functions of n variables

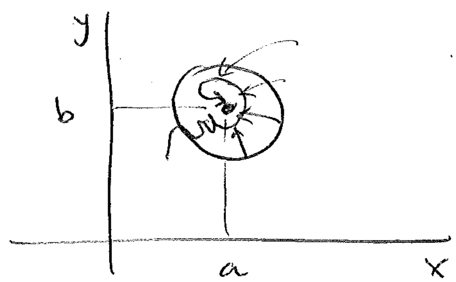
- First def. one sees:

We say $\lim_{x \to a} f(x) = L$ if

"$f(x)$ gets arbitrarily close to L

when x gets close to a".

Now, 2 variables



want to say:

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

if

ε measures this

"$f(x,y)$ gets arbitrarily close

to L as (x,y) gets

close to (a,b)".

<u>Def</u>: $\lim_{(x,y) \to (a,b)} f(x,y) = L$ if

$\forall \varepsilon > 0 \ \exists \delta$ s.t. a ball of radius δ

around (a,b) has some

points in the domain of f

<u>and</u> $\forall (x,y)$ in this ball

and in the domain

with $(x,y) \neq (a,b)$.

we have $|f(x,y) - L| < \varepsilon$.

Note: In $\mathbb{R}^n$,

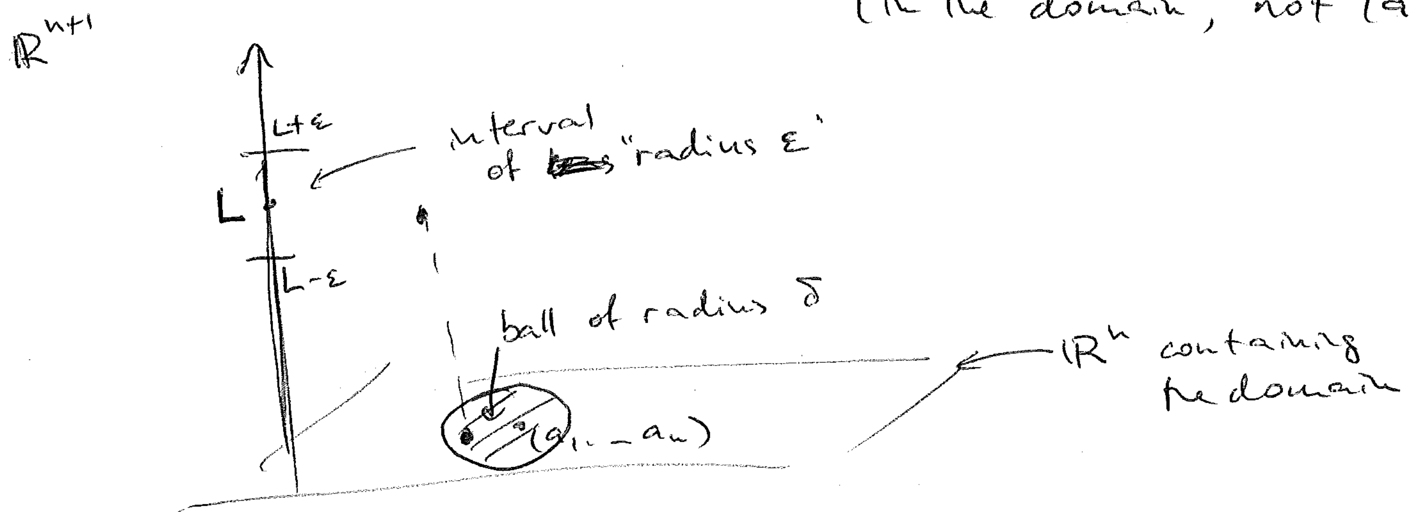
$$\lim_{(x_1 \ldots x_n) \to (a_1 \ldots a_n)} f(x_1 \ldots x_n) = L \quad \text{if}$$

. . . .

- the ball of radius δ lives in the domain so it is n-dimensional,

we still require $|f(x_1 \ldots x_n) - L| < \varepsilon$ for any $(x_1 \ldots x_n)$ in the ball.

(in the domain, not $(a_1 \ldots a_n)$).



Way to think about $\varepsilon - \delta$:

Suppose you are an engineer.

You have a boss who tells you how precisely your equipment needs to measure something complicated (e.g. volume of a box).

E.g. need to know the volume up to $0.001\,cm^3$.

This is the ε. (here $\varepsilon = 0.001\,cm^3$).

You have a tool that can measure length with good precision.

↑ in your control

This is your δ.

Your task is to figure out what δ to set so that the result really is within ε of the actual answer.

Example: You look at the box and think it is approximately $30\,cm \times 20\,cm \times 50\,cm$.

Want volume with precision $\varepsilon = 0.1\,cm^3$.

Suppose we know that the width is $30 \pm \delta$.

by magic we know the other dimensions are precisely 20 and 50 cm.

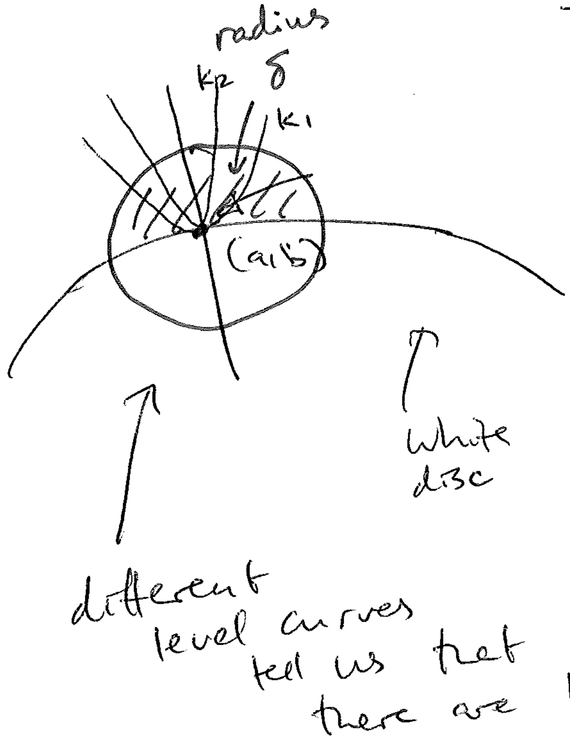
$$V = (30 \pm \delta) \cdot 20 \cdot 50\,cm^3 = 30 \cdot 10^3\,cm^3 \pm (1000\delta)\,cm^3.$$

for this box

$\boxed{\text{So need } \delta < \frac{\varepsilon}{1000}\text{ !}}$

$\boxed{\text{To make proof easier, Make } \delta < \frac{\varepsilon}{10{,}000}}$

Using contour plot to get information about limits (this refers to the picture linked separately).



Does the limit of $f(x,y)$ at (a,b) exist?

(at least two values: k_1 corr. to one ~~parabola~~, hyperbola k_2 corresp. to the other hyperbola)

$\boxed{\text{Fix } \varepsilon}$

If the limit existed, then for some δ,

suppose it was L

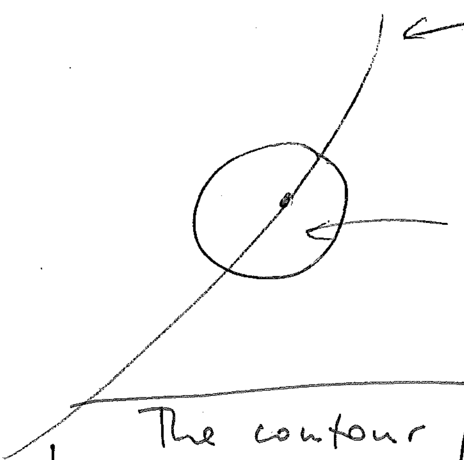
we'd have all values of f on this ball are within ε from L.

so we get $|k_1 - L| < \varepsilon$

$|k_2 - L| < \varepsilon$

Get: For any $\varepsilon > 0$, $|k_1 - L| < \varepsilon$, $|k_2 - L| < \varepsilon$ $\Rightarrow k_1 = k_2$!

Contradiction! So limit doesn't exist.



← level curve

here the function doesn't change much (otherwise we'd see a lot of other level curves there):

$$\boxed{\text{The contour plot \underline{suggests} (but does not \underline{prove}) that at such points the limit exists.}}$$

Ex: $\displaystyle \lim_{(x,y)\to(a,b)} \frac{\sqrt{x^2+y^2-4}}{3x+y}$

* Easy: plug in $(x,y) = (a,b)$ everywhere where both numerator and denominator are defined and $3x+y \neq 0$.

* Interesting: when cannot do this (e.g. when $3x+y=0$)

countour plot suggests, no limit as $(x,y) \to (a,b)$

↑ intersection of $x^2+y^2=4$ and $3x+y=0$

the "bad point" in the picture → $\boxed{(a,b) = \left(-\frac{\sqrt{2}}{\sqrt{5}}, +\frac{3\sqrt{2}}{5}\right) \text{ in our picture}}$

* homework: check algebraically that the limit exists at all other points of $x^2+y^2=4$.

* Read 12.2 and use $\varepsilon - \delta$: given ε, find δ.