Last time:

- Wanted to define cross product so that:
  
  \[ \tilde{u} \times \tilde{v} \perp \tilde{u}, \tilde{v} \] 
  + right-hand rule

  \[ |\tilde{u} \times \tilde{v}| = |\tilde{u}| |\tilde{v}| \sin \alpha \tilde{u} \] 

(1)

- Defined an algebraic operation

  \[ \tilde{u} \times \tilde{v} = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \] 
  \[ = \tilde{u} \times \tilde{v} \leq \text{as } \vec{m} \text{ (1)} \]

- Defined the operation for \( \vec{i}, \vec{j}, \vec{k} \).
- Used "distribution laws".

- Know: our operation \( \tilde{u} \times \tilde{v} \) has all these good properties, and satisfies
  
  \[ (\tilde{u} \times \tilde{v}) \cdot \tilde{w} = \text{volume of the box for any } \tilde{w} \]

- Really proved \[ \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \] 
  \[ \text{fact later this fall.} \]

- But we also proved using the parallelograms
  
  \[ (\tilde{u} \times \tilde{v}) \cdot \tilde{w} = \text{volume of the box.} \]

- Get: for any \( \tilde{w} \),
  
  \[ (\tilde{u} \times \tilde{v}) \cdot \tilde{w} = (\tilde{u} \times \tilde{v}) \cdot \tilde{w} \]

Then

\[ \tilde{u} \times \tilde{v} = \tilde{u} \times \tilde{v} \leq \text{so this is the cross product.} \]
Equations of Lines and Planes in \( \mathbb{R}^3 \)

* How to define a plane in \( \mathbb{R}^3 \).

Example: write an equation of the plane in \( \mathbb{R}^3 \) perpendicular to \( \vec{n} = \langle 1, 2, 3 \rangle \) and passing through \((0,5,4)\).

The vector \( \vec{r}' \) connecting a point \((x, y, z)\) in the plane with \((0,5,4)\) lies in the plane.

Then \( \vec{r}' \perp \vec{n} \).

Then \( \vec{n} \cdot \vec{r}' = 0 \). ← equation of a plane in vector form

\[ \vec{r}' = \langle x, y-5, z-4 \rangle \]

So, we get: \( \vec{n} \cdot \vec{r}' = 0 \)

\[ \langle 1, 2, 3 \rangle \cdot \langle x, y-5, z-4 \rangle = 0 \]

\[ x + 2(y-5) + 3(z-4) = 0 \]

\[ x + 2y + 3z = 22 \]

**Key:** to write an equation of a plane, you need to know its normal vector \( \vec{n} \), which was \( \vec{n} \) in the example.
All planes perpendicular to \( \overrightarrow{n} = \langle a, b, c \rangle \) have equations of the form

\[
ax + by + cz = d, \quad d \text{ is a constant.}
\]

The components of \( \overrightarrow{n} \).

(Note: It agrees with our example.)

The number \( d \) is determined by a point on the plane.

Example: Find equation of the plane passing through \( (0, 1, 2), \ (0, 0, 3), \ (1, 1, 1) \).

Solution: Need to find normal - call it \( \overrightarrow{n} \)

\( \overrightarrow{n} \) will be perpendicular to two vectors in our plane.

- Make two vectors from our points.
- Use cross product.

\[
\overrightarrow{v}_1 = -\overrightarrow{AB} = \langle 0, 1, -1 \rangle
\]

\[
\overrightarrow{v}_2 = \overrightarrow{BC} = \langle 1, 1, -2 \rangle
\]

\[
\overrightarrow{n} = \overrightarrow{v}_1 \times \overrightarrow{v}_2 = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  0 & 1 & -1 \\
  1 & 1 & -2
\end{vmatrix} = (-1)\mathbf{i} - \mathbf{j} - \mathbf{k}
\]

\[
= \langle -1, -1, -1 \rangle.
\]
Answer: \[-(x-0)-(y-1)-(z-2) = 0\]
\[x + (y-1) + (z-2) = 0\]

(using components of \(A\)).

Or:
\[x + y + z - 3 = 0\]
\[x+y+z = 3\]

Note: also gives us the area of
\[\Delta ABC: \ \frac{1}{2} | AB \times BC | = \frac{\sqrt{3}}{2}\]

What about lines:

1. two approaches: i) equation in terms of \((x,y,z)\)
   (like for the plane)
   not good for plotting!
   (easy to check if a given point satisfies it).

2. parametric equation: have parameter \(t\)
   for any value of \(t\) get a
   point on the line.
   (good for plotting it!)
Parametric equation of a line through A, B at \( t = 0 \) want to be at A at \( t = 1 \) want t be at B.

\[ \vec{r} = \langle x, y, z \rangle \] - vector from 0 to point \( (x, y, z) \) on the line.

Then \( \vec{r} = \vec{r}_0 + t \cdot \vec{r}_1 \) - parameteric equation of the line.

**Parametric equation in terms of coordinates**

\[ \langle x, y, z \rangle = \langle a_0, b_0, c_0 \rangle + t \langle a_1 - a_0, b_1 - b_0, c_1 - c_0 \rangle \]

\( A = (a_0, b_0, c_0) \)
\( B = (a_1, b_1, c_1) \)

\[
\begin{align*}
X &= a_0 + t(a_1 - a_0) \\
y &= b_0 + t(b_1 - b_0) \\
z &= c_0 + t(c_1 - c_0)
\end{align*}
\]

Next: symmetric equation

Read: 10.3 - done
10.4 - almost done
10.5 - home reading