Last: 1) boundary of a closed ball is a sphere.

will be important for continuity, differentiability.

Seek help: if not comfortable.

2) projections

- will review when we use them next week.

Today: Magic in $\mathbb{R}^3$:

cross product

Warm-up: mystery vector in $\mathbb{R}^3$.

- ask: give me the dot product of the mystery vector with the vector I give you.

Goal: guess the vector.

Easy: ask for

$$\bar{X} \cdot \bar{i} = a$$
$$\bar{X} \cdot \bar{j} = b$$
$$\bar{X} \cdot \bar{k} = c$$

$$\bar{X} = \langle a, b, c \rangle.$$

If $\bar{v}_1, \bar{v}_2, \bar{v}_3$ - given

know $\bar{X} \cdot \bar{v}_1$, $\bar{X} \cdot \bar{v}_2$, $\bar{X} \cdot v_3$ $\rightsquigarrow$ can recover $\bar{X}$,

if v_1, v_2, v_3 are linearly independent (not in the same plane).

1

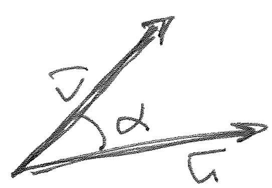
Def: $\bar{u}, \bar{v}$ - given vectors in $\mathbb{R}^3$.

Then $\bar{u} \times \bar{v}$ is a vector such that:

(1) is $\bar{0}$ if $\bar{u} \parallel \bar{v}$ (parallel)

is perpendicular to both $\bar{u}$ and $\bar{v}$ and $\bar{u}, \bar{v}, \bar{u} \times \bar{v}$ form a right-hand triple.

(2) $|\bar{u} \times \bar{v}| = |\bar{u}| \cdot |\bar{v}| \cdot \sin\alpha$



Notation: $\bar{u} \parallel \bar{v}$ means parallel ($\bar{u} = k \cdot \bar{v}$, $k \in \mathbb{R}$)

$\bar{u} \perp \bar{v}$ - perpendicular

(check: $\bar{u} \cdot \bar{v} = 0$).

- We need to show it exists!

- Is it unique? - Yes.

  - In other words,

(see next page)

Given $\bar{u}, \bar{v}$, can two different vectors satisfy the definition of $\bar{u} \times \bar{v}$?

– No: (1) defines direction (2) defines magnitude. uniquely.

Existence: will give a formula in terms of components of $\bar{u}, \bar{v}$.

First, some properties:

bilinear
$$\left\{\begin{aligned} t\bar{u} \times \bar{v} &= t \cdot (\bar{u} \times \bar{v}) \\ \bar{u} \times t\bar{v} &= t \cdot (\bar{u} \times \bar{v}) \\ \bar{u} \times (\bar{v}_1 + \bar{v}_2) &= \bar{u} \times \bar{v}_1 + \bar{u} \times \bar{v}_2 \\ (\bar{u}_1 + \bar{u}_2) \times \bar{v} &= \bar{u}_1 \times \bar{v} + \bar{u}_2 \times \bar{v} \end{aligned}\right.$$

– easy to check from def. (first two)

harder! (last two)

Note: $\bar{u} \times \bar{v} = -(\bar{v} \times \bar{u})$

- will construct an operation satisfying these properties, and then show it satisfies the def. of cross product.

- For $\bar{i}, \bar{j}, \bar{k}$, define

$$\boxed{\bar{i} \times \bar{j} = \bar{k} \qquad \bar{j} \times \bar{k} = \bar{i} \qquad \bar{k} \times \bar{i} = \bar{j}}$$

$$\begin{tikzcd} i \arrow[r] & j \arrow[r] & k \arrow[ll, bend right] \end{tikzcd}$$ – helps remember.

- Using bilinearity

$$\bar{u} = \langle a_1, b_1, c_1 \rangle = a_1 \bar{i} + b_1 \bar{j} + c_1 \bar{k}$$
$$\bar{v} = \langle a_2, b_2, c_2 \rangle = a_2 \bar{i} + b_2 \bar{j} + c_2 \bar{k}$$

$$\bar{u} \times \bar{v} = (a_1 \bar{i} + b_1 j + c_1 k) \times (a_2 i + b_2 j + c_2 k)$$

$$= \underbrace{a_1 a_2 \bar{i} \times \bar{i}}_{\bar{0}} + a_1 b_2 \underbrace{\bar{i} \times \bar{j}}_{k} + a_1 c_2 \underbrace{\bar{i} \times \bar{k}}_{-j}$$

$$+ b_1 a_2 \underbrace{\bar{j} \times \bar{i}}_{-k} + b_1 b_2 \underbrace{\bar{j} \times \bar{j}}_{0} + b_1 c_2 \underbrace{\bar{j} \times \bar{k}}_{i}$$

$$+ c_1 a_2 \underbrace{\bar{k} \times \bar{i}}_{j} + c_1 b_2 \underbrace{\bar{k} \times \bar{j}}_{-i} + c_1 \underbrace{c_2 \bar{k} \times \bar{k}}_{\bar{0}}$$

$$= \boxed{(b_1c_2 - c_1b_2)\bar{i} + (c_1a_2 - a_1c_2)\bar{j} + (a_1b_2 - b_1a_2)\bar{k}.}$$

- This is how we define $\bar{u}\times\bar{v}$ for $\bar{u} = \langle a_1, b_1, c_1\rangle$, $\bar{v} = \langle a_2, b_2, c_2\rangle$

Now, need to check that this vector satisfies the Definition.

Check: (1) from Def is easy: need to check this is perp. to $\bar{u}$, $\bar{v}$.

Compute dot products.

right-hand: a bit later.

(2) Magnitude: - will need to develop some theory in order to check that the magnitude is equal to $|\bar{u}||\bar{v}|\sin\alpha$.

- consider dot products of our $\bar{u}\times\bar{v}$ with an arbitrary vector $\bar{w}$. (mixed/triple product).

First, a short-hand for cross product.

$\bar{u} = \langle a_1, b_1, c_1\rangle$

$\bar{v} = \langle a_2, b_2, c_2\rangle$

$$\bar{u}\times\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \bar{i}\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - \bar{j}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \bar{k}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

easy to remember

$$\boxed{= \bar{i}(b_1c_2 - c_1b_2) - \bar{j}(a_1c_2 - c_1a_2) + \bar{k}(a_1b_2 - b_1a_2)}$$

Meaning of this determinant:

- when we replace the first row $\bar{i}, \bar{j}, \bar{k}$ in this determinant with 3 numbers – components of a vector $\bar{w} = \langle a_0, b_0, c_0 \rangle$

Get an honest determinant:

components of $\bar{w}$

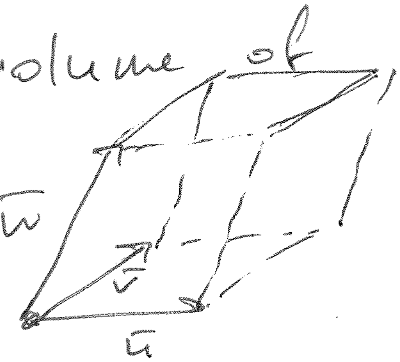
a number $\rightarrow$
$$\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = a_0 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_0 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_0 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

components of $\bar{u} \times \bar{v}$

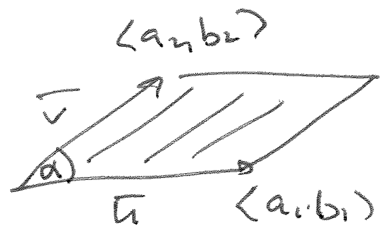
$$= \boxed{\bar{w} \cdot (\bar{u} \times \bar{v})}$$

mixed triple product of $\bar{w}, \bar{u}, \bar{v}$

- **Fact** $\bar{w} \cdot (\bar{u} \times \bar{v}) =$ volume of



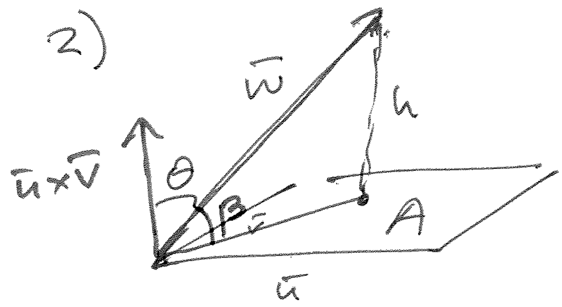
Proof: 1)



$$A = |a_1 b_2 - a_2 b_1| = \left\| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix} \right\|$$
absolute value of the det.

$$= |\bar{u}|\,|\bar{v}| \sin\alpha = |\bar{u} \times \bar{v}| \; (!)$$

2)



volume of box

$= h \cdot A$

$\shortparallel$

$|\bar{w}| \cdot \sin\beta = |\bar{w}| \cdot \cos\theta$

$\beta = \frac{\pi}{2} - \theta$

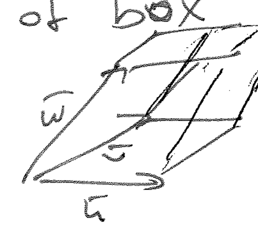
where θ = angle between $\bar{w}$ and $\bar{u} \times \bar{v}$

Then $h \cdot A = |\bar{w}| \cdot |\bar{u} \times \bar{v}| \cdot \cos\theta$

$= \bar{w} \cdot (\bar{u} \times \bar{v})$.

Punchline:

$$\underbrace{\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \bar{w} \cdot (\bar{u} \times \bar{v})}_{\text{general - works in } \mathbb{R}^n \text{ (later)}} = \text{volume of box}$$



So, we have proved:

- For the vector $\bar{u} \times \bar{v}$ that we defined, we always (for any w) have that:

  $\bar{w} \cdot (\bar{u} \times \bar{v})$ = volume of the box.

- For the vector $\bar{u} \times \bar{v}$ that we want in the geometric definition, we have $w \cdot (\bar{u} \times \bar{v})$ = volume of the box.

Then they match!